The sun revolves around the earth in an ellipse once every year

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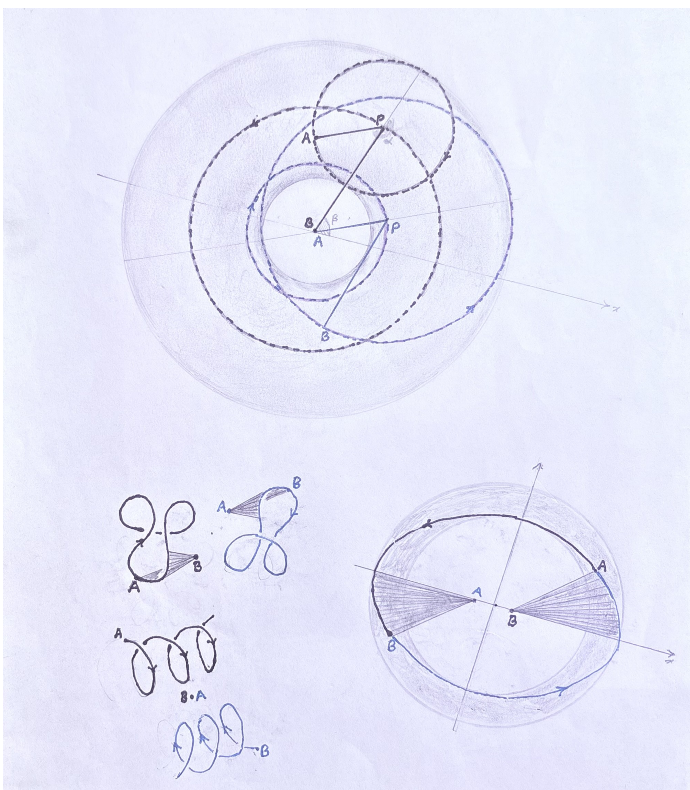
The en passant theorem of *calendars*, p.2, fn.4, I'd proved thus :- adding the *Ptolemy cycles* of that figure (or below with $BP = \frac{1}{r}, PA = 1$) gives $z = \frac{1}{r}e^{i\beta} + e^{i(\beta-\alpha)}$, so for $\alpha \equiv 2\beta$, $z = \frac{1}{r}e^{i\beta} + e^{-i\beta}$, i.e. $x = (\frac{1}{r}+1)\cos\beta, y = (\frac{1}{r}-1)\sin\beta$, an ellipse with semi-axes $\frac{1}{r} \pm 1$; also since each quarter of the ellipse is described in the same time, the radius vector from a focus does *not* sweep out equal areas in equal time; besides if the second cycle rotates backwards at any other constant angular speed this sum is not an ellipse (more below). □ But, if we don't insist (and seems Ptolemy did not always either) that the angular speed of a cycle be constant, then *any* periodic elliptic motion $z(t)$ is such a sum :- just define $\beta(t)$ so that $z(t) = \frac{1}{r}e^{i\beta(t)} + e^{-i\beta(t)}$. □

Since Ptolemy we've often regressed and now seem poised to proceed to zero! On 10/02/2026 I wrote *the sun revolves around the earth in an ellipse once every year* in three search engines; the bots that infest, patrol and now 'assist' in them all declared it incorrect and told me to interchange sun and earth; but of course the two statements are equally true or false, indeed *any motion of a point A relative to point B is congruent to that of B relative to A* :- any antipodal map of space preserves distances and interchanges the position vector at any time of A with respect to B with that of B with respect to A. □



Since orientation of space changes under an antipodal involution the *torsion* of a smooth motion of A relative to B is the negative of that of B relative to A. If the spatial motion is doodled planarly using bridges–bottom of figure–its inverse can be depicted by rotating it by 180 degress followed by changing all over/under crossings to under/over and vice versa, for example, *a right handed screw motion is inversely a left handed screw motion.* $\square$

A smooth motion with torsion always positive or negative has no *central symmetry,* i.e., an antipodal involution preserving it, but if there is one we can use the same curve to depict the inverse motion. $\square$

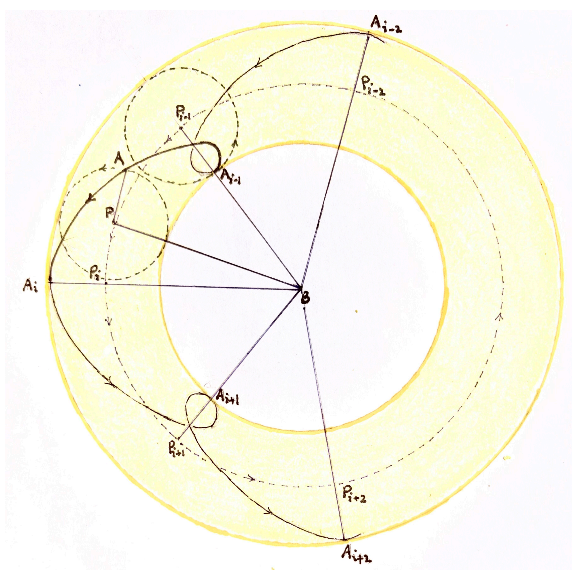
If B and A are trapped in a plane torsion is zero, there is an orientation preserving congruence with the inverse motion, and if there is a centre, rotating by 180 degrees around it we can use the same curve, e.g., the very same ellipse shows the inverse Kepler motion of the barycentre B of the sun relative to the barycenre A of *the earth at its other focus.* $\square$

A Ptolemy motion is any finite concatenation of steady circular motions: a point $A = P_0$ cycling around a point P_1 cycling around a point P_2 ... P_{i-1} cycling around a point $P_i = B$; even for 3 points A, P, B in a plane there is a vast variety of *Ptolemy curves* traced by A around B :-

Let $\underline{BP > PA}$ when these curves described by $\overrightarrow{BA} = \overrightarrow{BP} + \overrightarrow{PA}$ are clearly all in the *annulus* around B between radii $BP \pm PA$.

If the angular speed of P around B is zero we are reduced to A cycling around P fixed, so the curve is this circle, unless the angular speed of A around P is also zero, when it is merely a point of this circle. So from here on angular speed of P around B is nonzero, and we denote by $s \in \mathbb{R}$ how many times faster in the same or opposite direction A is cycling around P.

Theorem : *if s is rational the Ptolemy curve is closed, otherwise its closure is the entire annulus!*



If $s = 0$, i.e., the angular speed of A around P is zero, $\angle APB$ remains constant, and depending on its value, the curve traced by A can be any one of the concentric circles of the annulus.

If $s \neq 0$ this angle varies at a constant rate, so there exist successive collinear positions (B, A_i, P_i) or (B, P_i, A_i) giving, alternately, *all points A_i of the curve on the inner and outer boundary* of the annulus; on each *part $A_{i-1} \rightsquigarrow A_i$ of our curve* the radial distance is strictly increasing or decreasing; and as shown in figure the next arc $A_i \rightsquigarrow A_{i+i}$ *is its reflection in the line BA_i.*

Since a half rotation around P takes A from A_{i-1} to A_i, *the point P moves on the axial circle from P_{i-1} to P_i by $\frac{1}{2s}$th of a full rotation around B.* If this sequence P_i repeats there is a minimum positive and *even*—because A_i has to return to the same, inner or outer, boundary circle—number of steps or *period* $2p$ after which $P_i = P_{i+2p}$ and $\frac{2p}{2s} = q$, this being the number of full rotations around B made in this process: so $s = \frac{p}{q}$ is rational. If s is irrational, the points P_i are all distinct, and within any given positive distance, of any of them, there is another, so they form an infinite dense subset of the axial circle: which implies the result using the last paragraph. *q.e.d.*

We denote now by $\beta(t) \in \mathbb{R}$ the angle from a chosen x-axis to the direction at time t of $\overrightarrow{BP}$ cycling around B and by $\alpha(t) \in \mathbb{R}$ the further angle from this rotating direction to the direction then of $\overrightarrow{PA}$ cycling around P. So $s \in \mathbb{R}$ *is the constant such that* $\alpha(t) \equiv s\beta(t)$ and only this identity matters[1] for the curve traced by A, which we'll parametrize by $\beta \in \mathbb{R}$.

To work out for $s \neq 0$ *the geometry of Ptolemy curves near their points on the boundary circles* it is convenient to choose an x-axis in one of these direction BA_i; also to think of the plane as $\mathbb{C}$ with this its real axis, and imaginary axis its rotation forward around B by $\pi/2$; so $\overrightarrow{BA}(\beta) = BP(e^{i\beta} \pm re^{i(\beta+\alpha)}) = \mathrm{BP}e^{i\beta}(1 \pm re^{is\beta}), r := \frac{PA}{BP}$, the sign depending on whether our x-axis passes through a point A_i on the outer or the inner circle:-

The derivative of $e^{i\beta} \pm re^{i(1+s)\beta}$ being $ie^{i\beta}(1 \pm r(1+s)e^{is\beta})$ *all contacts with outer and inner circles are tangential unless* $r = -\frac{1}{1+s}$ resp. $r = \frac{1}{1+s}$; also, since $r \in (0, 1)$, these exceptions arise iff $s < -2$, resp. $s > 0$; and then there is a *cusp* tangential to BA_i since Δy and Δx are of order β^3 and β^2 near A_i.

Doodled next are a few curves for rational s showing in particular how cusps and then *retrograde loops* touching the outer or inner circle emerge as $r \in (0, 1)$ increases so that $1 + r + rs$ or $1 - r - rs$ changes sign. For s near 0 curves rotate around the annulus repeatedly, so there are many *self-intersections* even if there are no local loops. I've prioritized understanding over accuracy and app-adept readers can doodle even *charming videos of their creation*? But no hand or app can expose *the most important Ptolemy curves, that occur with probability one,* viz., those with s an irrational: they go around and around forever without closing and are dense in the entire annulus.

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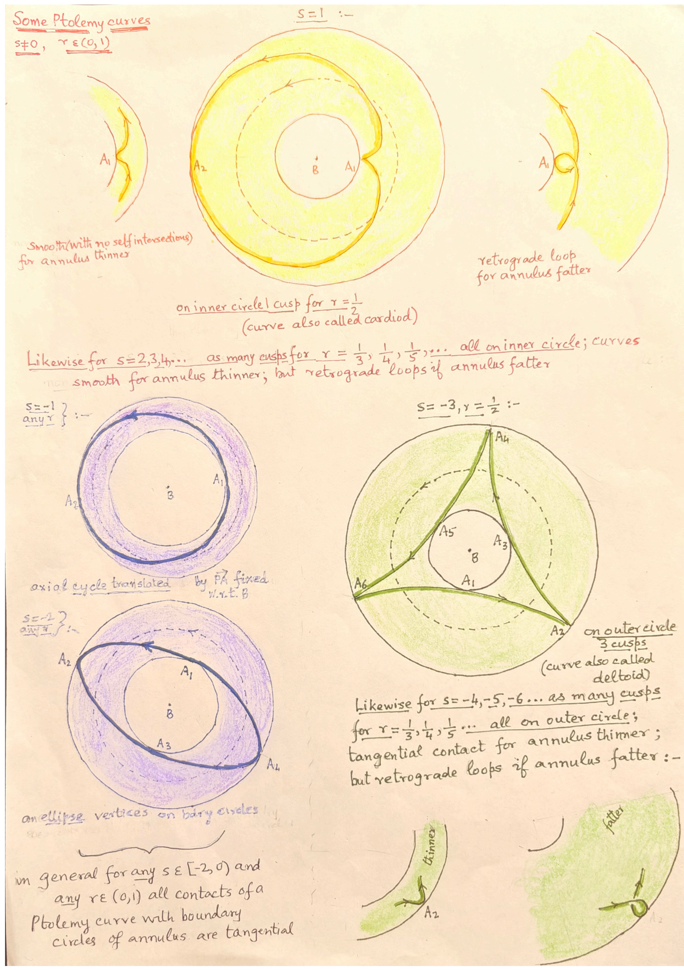
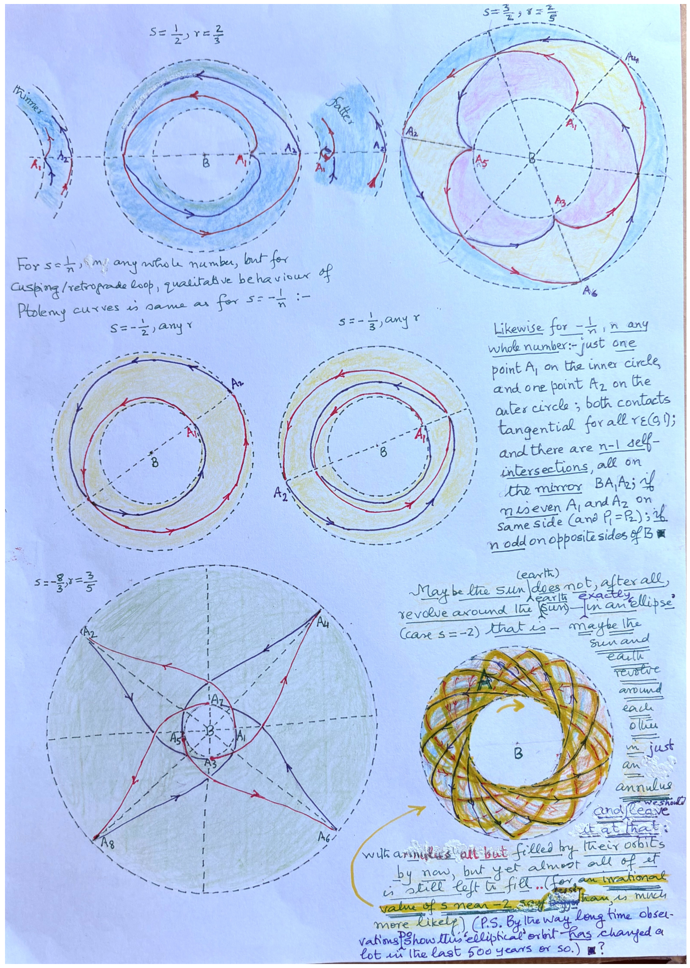
[1] So <u>theorem</u>: *the same Ptolemy curve or a portion of it is traced even if the two circular motions are arbitrary in time as long as they are tied by $\alpha(t) \equiv s.\beta(t)$.* □ Cf. first para, but note that case, A turning around P twice as fast backwards, corresponds to $\alpha \equiv -2\beta$ because we are now using *signed* angles, now $\alpha \equiv 2\beta$ signifies twice as fast forwards.



Only the principles used in the proof of the main theorem were used. As r increases from 0 (axial forward cycle) towards 1 annulus thickens to a disk and depending on whether the smaller cycle is also forward or not a cusp develops at a specific thickness on the inner or outer circle–unless we're in the *smooth regime* from ellipse $s = -2$ till 0–after which it converts to a local loop. The first lot above show what curves are like for $s \neq 0$ an integer, the second for their reciprocals, and some other fractions beauty of shape because self-intersections must be on the mirrors that generate any curve from a single *riser,* its DNA so to speak, the bit from a point on the inner circle to the point next on the outer circle. We conclude with an impressionistic doodle of what the curve looks like globally for s *an irrational slightly less than* $s = -2$ so this DNA is but slightly short of a quarter-ellipse but what it develops now into as an adult is totally different, which should immediately suggest to any reasonable mind the earth and sun in fact do not move in an ellipse around each other.



And oh yes that $BP > PA$ was not essential: an antipodal reflection of space preserves the annulus only reverses its chosen forward orientation. Tthe curves for $r > 1$ are very much in the same annulus, and if we switch A and B tels us how A sees motion of B. Note too that along with r the parameter s has to switched to its reciprocal. And if if we desingularize the curves in the obvious way to toral motion corresponds to the involution of its longitude and latitude, using which two solid tori pasted together give us the 3-sphere. For $r \to 1$, the annulus becomes a disk,and our curves have interesting limits, for example, that ellipse becomes a segment.

On the formal side the differential closure of these curves–add all other solutions too of the simple differential equations obeyed–gives us many more curves, for example from ellipse all conic sections, and complexification yet more; when you'll start musing with Hamilton of Kepler on the unit 3-sphere of quaternions. However what intrigues me most (as you must have gathered from this hurried writing this is very much TBC here) is the simplification of all this by desingularization and covering spaces to just straight line motion of slope s on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ – the generalization to finitely many cycles is obvious – when we indeed are but pictorially at the euclidean or continued fraction natural defining expansion of any positive real number s. For my mind has wandered back now 1974 when I saw that some differential topological invariants, associated to all smooth foliations, are, for the case of these toral linear irrational foliations finite dimensional or not depending on how fast the whole numbers in their continued fractions expansion is increasing …