

Tangent cones

6. The notion of right derivatives generalizes to **forward derivatives** $f'_u(x)$ along a vector $u \neq 0$, of continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of n variables, viz., all¹ limits of the difference quotient $\frac{f(x+hu)-f(x)}{h}$ as $h > 0$ tends to zero. The ring of all such functions $C^0(\mathbb{R}^n)$ is basic because a bijection of \mathbb{R}^n is bicontinuous, that is a *homeomorphism, if and only if it induces an isomorphism of $C^0(\mathbb{R}^n)$* :- to see 'if' recall this topology is generated by all open affine half spaces, i.e., inverse images of infinite open intervals under linear surjections $\mathbb{R}^n \rightarrow \mathbb{R}$. \square In general $f'_u(x)$ is singleton or a closed subinterval of $[-\infty, +\infty]$ ² :- if liminf and limsup are distinct, using continuity in h , we can find a sequence $h_i > 0$ approaching zero with all $\frac{f(x+h_i u)-f(x)}{h_i}$ equal to any given intermediate value. \square We shall now put additional conditions on forward derivatives to define some more rings³, and for each there shall be the corresponding subgroup of homeomorphisms inducing an isomorphism of this ring.

The mildest punctual condition that suggests itself is that *forward derivatives be finite*⁴, i.e., that each $f'_u(x)$ be a closed subinterval $[a, b]$ (possibly $a = b$) of $(-\infty, +\infty)$. The same *direction* is indicated by any positive scalar multiple of $u \neq 0$, and $f'_{ku}(x) = k f'_u(x)$ for $k > 0$, so *at each x over each direction* there is a *forward cone of rays* tangent to f . Such functions form a *subring of $C^0(\mathbb{R}^n)$* :- For, almost the same arguments show all forward derivatives of $f + g$ and fg are still given by the usual *sum and product rules*, viz., $(f + g)'_u(x) = f'_u(x) + g'_u(x)$ and $(fg)'_u(x) = f'_u(x)g(x) + f(x)g'_u(x)$, using here the obvious continuous extension of the sum and product of real numbers to the **space $\mathcal{I}(\mathbb{R}) \supset \mathbb{R}$ of all closed intervals** of real numbers. \square Note $\mathcal{I}(\mathbb{R})$ is homeomorphic to a closed half-plane, viz., the quotient of $\mathbb{R} \times \mathbb{R}$ under $(a, b) \leftrightarrow (b, a)$; so *all finite open intervals of reals $\mathcal{H}(\mathbb{R})$ to an open half-plane*, viz., the quotient of the *deleted product* $\mathbb{R} \times \mathbb{R} \setminus$ diagonal (singletons) under this involution.

So, we extend to the ring $C^{0\ell}(\mathbb{R}^n)$ of all continuous functions $F : \mathbb{R}^n \rightarrow \mathcal{I}(\mathbb{R})$ the notion of forward derivatives $F'_u(x)$:- viz., closed subinterval of $[-\infty, +\infty]$ of points such that any neighbourhood intersects finite interval $\frac{F(x+hu)-F(x)}{h}$ for some $h > 0$ arbitrarily close to zero; continuity in h of this $\mathcal{I}(\mathbb{R})$ -valued difference quotient shows that values intermediate between two such limits are indeed also limits. \square We define its subring $C^{\frac{1}{2}\ell}(\mathbb{R}^n)$ by the condition that all $F'_u(x)$ be finite, then $C^{1\ell}(\mathbb{R}^n)$ by their continuity in x ; then ask for finiteness of all forward second derivatives $F''_{vu}(x)$, then their continuity in x ; next finiteness of forward third derivatives $F'''_{wvu}(x)$ and so on; these progressively stronger conditions defining an infinite train of smaller and smaller subrings.

$C^{\frac{1}{2}}(\mathbb{R}^n) \supset C^1(\mathbb{R}^n) \supset C^{1\frac{1}{2}}(\mathbb{R}^n) \supset C^2(\mathbb{R}^n) \supset \dots$ will denote the similar train

¹this natural generalization is imho the key to a simple manifold theory!

²or **Th.1** : to visualize the given proof draw a suitable figure say like that in 'Distances and homeomorphisms' (2024). That the intermediate value theorem is used in it had led here to a detour to Poincaré's work on the 3-body problem!

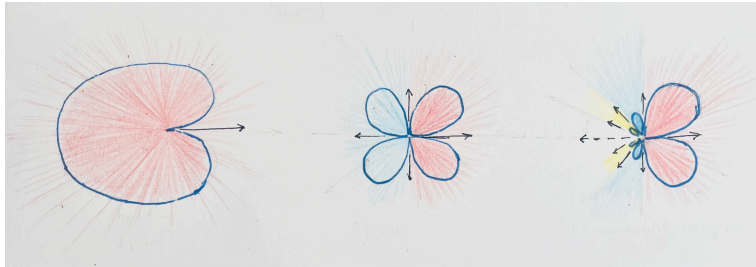
³or subsets if closure w.r.t. addition & multiplication is in doubt.

⁴are there closed 4-manifolds so wild that it is not meaningful to speak even of globally defined continuous functions satisfying this condition?

under the rather stringent condition that forward derivatives be single valued:- $C^{\frac{1}{2}}(\mathbb{R}^n) \subset C^{\frac{1}{2}\ell}(\mathbb{R}^n)$ consists of all $f \in C^0(\mathbb{R}^n)$ with a unique finite forward derivative at any x along any nonzero vector u , i.e., a *unique tangent ray over any direction* at any x ; if further all $f'_u : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous that is $C^1(\mathbb{R})$; then if each of these has a unique finite forward derivative $f''_{vu}(x)$ at any x along any nonzero vector v that makes $C^{1\frac{1}{2}}(\mathbb{R}^n)$; and if all these second derivatives $f''_{vu} : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous that gives $C^2(\mathbb{R}^n)$; and so on. For example, if the restrictions of $f \in C^0(\mathbb{R}^n)$ to the cells of a locally finite *polytopal subdivision* of \mathbb{R}^n extend to functions of $C^1(\mathbb{R}^n)$ then $f \in C^{\frac{1}{2}}(\mathbb{R}^n)$; likewise $C^{r\frac{1}{2}}(\mathbb{R}^n)$ contains r times continuously differentiable functions whose restrictions to the cells of any triangulation are differentiable once more. This because a small initial open interval of any ray from a point on a cell is either in it or outside it; so also an $f \in C^0(\mathbb{R}^n)$ is in $C^{\frac{1}{2}\ell}(\mathbb{R}^n)$, respectively $C^{\frac{1}{2}}(\mathbb{R}^n)$, iff its restriction to each cell has finite forward derivatives, respectively a unique finite forward derivative, along each direction initially in the cell. \square

The **tangent cone** at x of $f \in C^{\frac{1}{2}\ell}(\mathbb{R}^n)$ is the graph in $\mathbb{R}^n \times \mathbb{R}$ of $u \mapsto f'_u(x)$ (here $f'_0(x) := 0$), i.e., the union of rays from the origin of \mathbb{R}^{n+1} through all $(u, f'_u(x)) \subset S^{n-1} \times \mathbb{R}$ as u runs over the unit sphere S^{n-1} of \mathbb{R}^n . For $n \geq 2$ the stronger punctual condition, *tangent cones unbroken*, i.e., directional derivatives $f'_u(x) \in \mathcal{I}(\mathbb{R})$, $u \in S^{n-1}$ continuous in u , gives *smaller rings* $C^{\frac{3}{4}\ell}(\mathbb{R}^n)$, $C^{\frac{3}{4}}(\mathbb{R}^n)$; a lot of these cones can be truly conical, yet this seems exceptional: it seems all tangent cones, except on a subset of points $x \in \mathbb{R}^n$ of measure zero, consist of linear subspaces of \mathbb{R}^{n+1} . Anyway, the obvious localization of the last condition wipes out all exceptions: If $f \in C^{1\ell}(\mathbb{R}^n)$, i.e., if $f'_u(x) \in \mathcal{I}(\mathbb{R})$ are continuous in x , they are necessarily continuous also with respect to u , and now *each tangent cone is a union of codimension one subspaces of \mathbb{R}^{n+1}* :-

This because, if $f \in C^1(\mathbb{R}^n)$ the unique continuous forward derivative along any linear combination $u = c_1e_1 + \dots + c_n e_n$ of a vector space basis is given by the *chain rule* $f'_u(x) = c_1 f'_{e_1}(x) + \dots + c_n f'_{e_n}(x)$ —or $\frac{d}{dt} f(x_1 + tc_1, \dots, x_n + tc_n) = c_1 \frac{\partial f}{\partial x_1}(x_1, \dots, x_n) + \dots + c_n \frac{\partial f}{\partial x_n}(x_1, \dots, x_n)$ for usual basis—and a like argument (this shall be recalled in more detail a bit later below) shows this rule is valid even for interval-valued functions $f \in C^{1\ell}(\mathbb{R}^n)$. (We won't insist on capitals for interval-valued functions, and will often use $C^{1\ell}(\mathbb{R}^n)$ etc. for $C^0(\mathbb{R}^n) \cap C^{1\ell}(\mathbb{R}^n)$ etc. too, though these may well be not much bigger than $C^1(\mathbb{R}^n)$ etc.)



On the other hand, consider the region of \mathbb{R}^n , $n \geq 2$ obtained by rotating a *heart* around the tangent from its cusp, and define $f : \mathbb{R}^n \rightarrow \mathbb{R}$ to be zero on

this ray, while on a point of any other ray from cusp the value of f equals the distance from it until its exit from heart, after which it becomes constant; this gives us an $f \in C^{\frac{1}{2}}(\mathbb{R}^n)$ with directional derivative at cusp 0 along the forward tangent ray, but in all other directions it is 1. \square

More generally, consider a flower with *many petals* assigned distinct $a_i > 0$, then we can define an $f \in C^{\frac{1}{2}}$ which is 0 on forward tangent rays at the cusp, and in case there are *infinitely many petals* (there are flowers with even uncountably many) also on their limit rays—near these rapidly damp out the petal size—and along a ray through the i th petal f is a_i times the distance from cusp till exit, after which it becomes constant; so the directional derivatives of f at the cusp along these directions are a_i . \square

So finiteness of each $f'_u(x)$ does not imply $\{f'_u(x), u \in S^{n-1}\} \subset \mathbb{R}$ is bounded, but it is so *if each $f'_u(x)$ is bounded on a neighbourhood of x* :-

Indeed, it suffices to assume that the n directional derivatives $\frac{\partial f}{\partial y_i}$ along the usual basis vectors $e_i = (0, \dots, 0, 1, 0 \dots 0)$ are so bounded, equivalently that the n difference quotients $\frac{f(y_1, \dots, y_i+t, \dots, y_n) - f(y_1, \dots, y_i, \dots, y_n)}{t}$, $t > 0$ small, are bounded for all $y = (y_1, \dots, y_n)$ in some neighbourhood of x . So, since any unit vector u is a linear combination $c_1 e_1 + \dots + c_n e_n$ with all $|c_i| \leq 1$, and the difference quotient at x in this direction $\frac{f(x_1+c_1t, \dots, x_n+c_nt) - f(x_1, \dots, x_n)}{t}$ can be rewritten as $\sum \pm c_i \frac{f(x_1, \dots, x_{i-1}, x_i+c_it, \dots, x_n+c_nt) - f(x_1, \dots, x_{i-1}, x_i, x_{i+1}+c_{i+1}t, \dots, x_n+c_nt)}{|c_i|t}$ over all i such that $c_i \neq 0$, we see that $\{f'_u(x), u \in S^{n-1}\}$ is bounded. \square

This rewriting is also the main point in the argument for the chain rule, that implied the stronger conclusion that $f'_u(x)$ is continuous with respect to u , under the hypothesis that $\frac{\partial f}{\partial y_i}$ are continuous in some neighbourhood of x ; the heart example shows this stronger conclusion is invalid under the weaker hypothesis above; and above result shows that, for the example with infinitely many petals with $a_i > 0$ unbounded, even this weaker hypothesis must be invalid.

If *distortion of distance is bounded* so are directional derivatives; and, under *convexity*, also conversely:- i.e., $|f(y) - f(x)| < c|y - x| \forall x, y \in U$ open, implies all $\frac{f(x+ut) - f(x)}{t}$, $u \in S^{n-1}$, $t > 0$ small, and so the closed intervals $f'_u(x)$ of their limits, are contained in $(-c, +c)$; conversely $f'_u(x) \subset (-c, +c)$ gives $|f(x_1) - f(x)| < c|x_1 - x|$ if x_1 is on the ray from x in direction u and sufficiently close to x , from which—use triangle inequality and compactness of a segment—it follows that $|f(y) - f(x)|$ is less than c times the length of any broken-line $[x_0, x_1, \dots, x_t]$ in U from $x = x_0$ to $x_t = y$. \square But the weaker conclusion that distortion of distance under f is locally bounded involves no convexity and follows also from the boundedness of any n basic directional derivatives in that neighbourhood by using once again rewriting above. \square

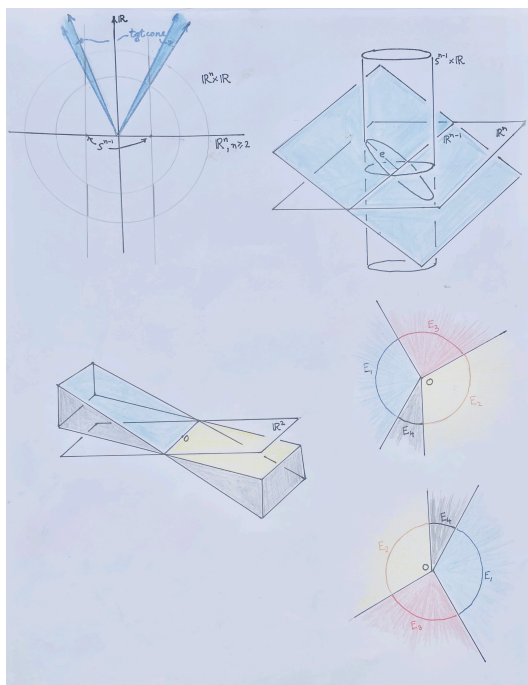
So, it is the lipschitz homeomorphisms of \mathbb{R}^n that induce isomorphisms of this ring $C^L(\mathbb{R}^n)$ of functions with directional directives locally bounded:- Since distortion of distance under such a homeomorphism h and its inverse is locally bounded $f \in C^L(\mathbb{R}^n) \iff fh \in C^L(\mathbb{R}^n)$. Now take a homeomorphism such that, in some locality, for any $c_i \rightarrow \infty$, there is a pair of points x_i, y_i such that the distance between the associated pair $|y'_i - x'_i| > c_i|y_i - x_i|$. Then its

composition with the orthogonal projection p_i on the line through the origin parallel to x'_i, y'_i gives a function f_i such that $|f_i(y_i) - f_i(x_i)| > c_i|y_i - x_i|$. So composition with a limit p of these p_i is a function $f \notin C^L(\mathbb{R}^n)$. \square

Consider *differences of all pairs of directional derivatives bounded*; punctually it is the same as, all directional derivatives bounded, for they'll be no farther than this bound from any of them; but *it is a weaker local condition*, and holds if just the $\binom{n+1}{2}$ *sets of differences* of directional derivatives along any orthogonal basis, say *all* $\frac{\partial f}{\partial y_i} - \frac{\partial f}{\partial y_j}$ *are bounded over* U :- use the same rewriting as for the analogous result for directional derivatives locally bounded. \square

Using distances $r \geq 1$ from the origin of $\mathbb{R}^n \times \mathbb{R}$ on the **unit tangent cones** of $f \in C^{\frac{1}{2}\ell}(\mathbb{R}^n)$, i.e., the parts of the tangent cones covering the unit sphere $S^{n-1} \subset \mathbb{R}^n$, the local boundedness of directional derivatives is equivalent to saying that $\sup(r)$ is bounded on unit tangent cones at all points of the given open $U \subset \mathbb{R}^n$, while *local boundedness of differences of directional derivatives is, for $n \geq 2$, equivalent to saying that the **eccentricity** $\sup(r) \div \inf(r)$ of unit tangent cones is locally bounded, and a bit more*, that we'll now formulate:- A unit tangent cone lies on the portion of the cylinder $S^{n-1} \times \mathbb{R}$ intercepted by the annulus between n -spheres with the origin as centre and radii $\sup(r)$ and $\inf(r)$. This condition is new iff *all directional derivatives stay away from zero*, i.e. $\inf(r) > 1$, then the cylindrical intercept has two components, one above and the other below \mathbb{R}^n . If differences of derivatives is bounded, *at points where $\sup(r)$ is big enough a tangent cone points wholly upwards or wholly downwards*, though this can vary from point to point of U . Conversely this and eccentricity bounded implies differences of derivatives bounded. \square

The first doodle below is of such *an upward cone* at $x \in U$; rather its planar section covering a line L of $\mathbb{R}^n, n \geq 2$ through the origin; so tangent rays fill the closed intervals $f'_{\pm\lambda}(x) \in \mathcal{I}(\mathbb{R})$ above $\{\pm\lambda\} = S^{n-1} \cap L$; and the two circles drawn with the origin as centre pass through the highest and lowest point of their union; so $\sup(r)$ and $\inf(r)$ are respectively the sup and inf of their radii over all L ; but maybe this section varies discontinuously [?] with L .



For $f \in C^1(\mathbb{R}^n)$, i.e., finite unique and continuous directional derivatives, a tangent cone is a codimension one subspace of $\mathbb{R}^n \times \mathbb{R}$; cutting \mathbb{R}^n in an \mathbb{R}^{n-1} iff all derivatives are not zero; so it cuts $S^{n-1} \times \mathbb{R}$ in an *ellipsoid*-case $n = 2$ is drawn—containing an $S^{n-2} \subset S^{n-1}$; so $\inf(r) = 1$ and $\sup(r) = e$ its sole semi-axis > 1 , justifying why their ratio is called eccentricity.

If we drop unique, i.e., if $f \in C^{1\ell}(\mathbb{R}^n) \supset C^1(\mathbb{R}^n)$, a tangent cone can be $n+1$ dimensional, but we saw that it is still a union of codimension one subspaces, which constrains how the intervals of directional derivatives $f'_u(x) \in \mathcal{I}(\mathbb{R})$ vary with direction. For starters if $f'_u(x) = [a, b]$ then $f'_{-u}(x) = [-b, -a]$, and more generally knowing these intervals in linearly independent directions determines them in their linear span. In particular *there a maximal subspace $\mathbb{R}^{n-i} \subset \mathbb{R}^n$ in which directions derivatives are unique and the tangent cone is supported by 2^i hyperplanes containing \mathbb{R}^{n-i} and one of the two ends of the intervals $f'_u(x)$ in any i linearly independent transverse directions.* \square

Also drawn for $n = 2$ are two such *solid cones* supported by 2^1 and 2^2 planes; for the former, the depicted unique directional derivatives are 0, but can be any antipodal pair $\pm a$ of real numbers; for the latter, only the top and bottom *fans* made by the four supporting planes are drawn, with the antipodal pairs of arcs of the four ellipses in which they intersect the cylinder.

We recall that a C^1 -manifold can be *smoothed* uniquely to a C^∞ -manifold and there is a nice obstruction and classification theory of such manifolds: is this class the same as of all those *on which it makes sense to speak of locally* ⁵

⁵This phrase always as spelled out by de Rham on the first three pages of his beautiful

$C^{1\ell}$ -functions and what about even milder notions of differentiability?

Returning to C^Q -functions on \mathbb{R}^n , i.e., with differences of derivatives locally bounded, equivalently eccentricity of unit tangent cones locally bounded, here for starters is its finitistic reformulation. We want the *oscillation* $|f(x_1) - f(x_2)|$ divided by radius bounded over n -balls with centres x ; equivalently the *segment* P_1P_2 , where $P_1, P_2 \in \mathbb{R}^n \times \mathbb{R}$ are above $x_1, x_2 \in \mathbb{R}^n$ on $\text{graph}(f)$, divided by radius, is bounded:- think of the origin O variably as $P = (x, f(x))$. \square Or, using *angle*, i.e., great circle distance on unit sphere between rays—so angles run from 0 through $\frac{1}{2}|S^1| = \pi$ - *excess* $\angle P_1PP_2 - \angle x_1xx_2$ should be bounded by a constant less than π ; or else, its infinitesimal reformulation:- Now $P = O$ and P_1, P_2 are on the unit tangent cone above $x_1, x_2 \in S^{n-1}$ and this condition (it kicks in only if x_1 and x_2 are close) is equivalent to the boundedness of P_1P_2 because the *angle subtended* $\angle P_1P_2$ at O by this segment is maximum when its centre is on \mathbb{R}^n and decreases towards zero as it slides up or down on the vertical plane (or vertical line if $x_1 = x_2$) containing P_1 and P_2 . \square Or, in terms of the intercepts on the unit sphere S^n of $\mathbb{R}^n \times \mathbb{R}$ with equator S^{n-1} , we want differences of latitudes bounded by a constant less than π ; etc.

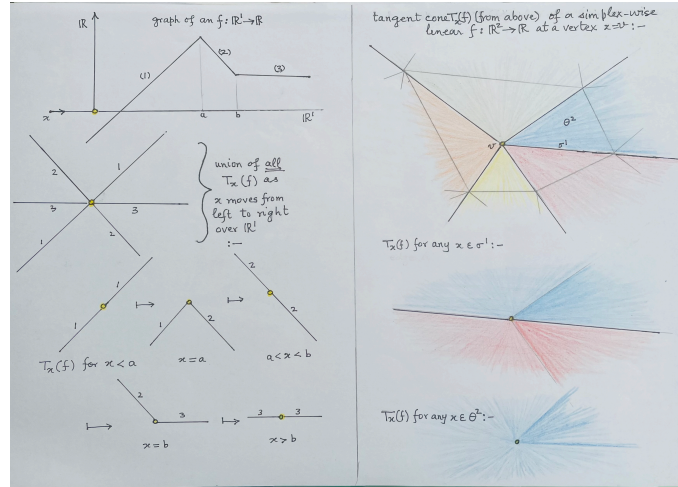
Indeed, if we demand that all i -simplices $P_1P_2 \dots P_{i+1}$ with vertices on unit tangent cones - so $x_1x_2 \dots x_{i+1}$ have dimension i or $i - 1$ - of an $f \in C^{\frac{1}{2}\ell}(\mathbb{R}^n)$ have *i -volumes* locally bounded, this gives us milder and yet milder notions of differentiability, that is, we get a strictly decreasing train of rings $C^{\frac{1}{2}\ell}(\mathbb{R}^n) \supset C^{Q^n}(\mathbb{R}^n) \supset \dots \supset C^{Q^2}(\mathbb{R}^n) \supset C^Q(\mathbb{R}^n) \supset C^L(\mathbb{R}^n) \supset C^{1\ell}(\mathbb{R}^n) \supset \dots$ and we can reformulate these notions using *i -angles* $\angle P_1P_2 \dots P_{i+1}$, i.e., the i -volumes intercepted on S^n by the cones over our i -simplices, so these are at most $\frac{1}{2}|S^i|$ which is the case iff O is in the i -simplex. Equivalently the excess $\angle P_1P_2 \dots P_{i+1} - \angle x_1x_2 \dots x_{i+1}$ over the i -angle subtended by the projected (possibly degenerate) i -simplex is bounded by a constant less than $\frac{1}{2}|S^i|$. This again because $\angle P_1P_2 \dots P_{i+1}$ varies inversely with the height above or below \mathbb{R}^n of the barycentre of our simplex. \square

These classes C^{any} of functions make sense even on a *manifold* M^n covered with *charts* related by *transformations* $\mathbb{R}^n \supset U \leftrightarrow V \subset \mathbb{R}^n$ with all n coordinates such functions:- these bijections form a *pseudogroup*. \square It seems any manifold has a *structure* of class $C^{\frac{1}{2}\ell}$ i.e. a maximal covering by charts interrelated thus; which contains for $n \neq 4$ a unique maximal covering by charts related in a C^L manner; but some 4-manifolds do not admit even a C^{Q^n} structure? The answer seems tied to $2+2 = 2 \times 2$ duality and a 3-dimensional cohomology class seminal for structures of the *piecewise linear* class C^{PL} ...

So let's consider *how* $C^{PL}(\mathbb{R}^n)$ nests between $C^{\frac{3}{4}}(\mathbb{R}^n)$ and $C^1(\mathbb{R}^n)$:- We recall $C^{\frac{3}{4}\ell}(\mathbb{R}^n)$ -respectively $C^{\frac{3}{4}}(\mathbb{R}^n)$ -denotes the set of all functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with directional derivatives $f'_u(x)$ finite-respectively finite and single valued-and for each $x \in \mathbb{R}^n$ continuous in u , i.e., whose tangent cones $T_x(f)$ are *unbroken* over the unit sphere $S^{n-1} \subset \mathbb{R}^n$. If $f \in C^{PL}(\mathbb{R}^n)$, i.e. its restrictions to the simplices of a locally finite triangulation of \mathbb{R}^n are linear, $T_x(f)$ has as its rays from the

book *Variétés différentiables* (1960); also not bad at all is my paper, *From calculus to cyclic cohomology* (1995), for what led to this book and beyond.

origin O of $\mathbb{R}^n \times \mathbb{R}$ parallels to the rays from $(x, f(x))$ which extend the segments of $\text{graph}(f)$ over all $[x, y], y \in Lk(x)$.



So $T_x(f)$ is an unbroken polytopal cone at O whose facets are parallel to the f -images of the n -simplices incident to x , and for a generic f these facets are distinct. Thus $T_x(f)$ is an n -plane if x is in the interior of an n -simplex but is usually wrinkly otherwise. Also all nearby cones are glued to $T_x(f)$: if we move from x to a sufficiently close x' it is in the same simplex or an incident higher dimensional simplex and $T_{x'}(f)$ contains the facets of $T_x(f)$ corresponding to n -simplices that are also incident to x' .

The wrinkliness of cones arises from the discontinuities in x of the directional derivatives $f'_u(x), u \in S^{n-1}$, viz., the jumps of slope of the piecewise linear f along all directed lines parallel to u . However since these jumps can occur only where the directed line goes through a lower dimensional simplex, we see that if $y \in \mathbb{R}^n$ approaches x remaining always in a closed cone $\phi^n(x)$ extending some incident closed n -simplex θ^n then we do have $\lim f'_u(y) = f'_u(x)$ for all directions $u \in S^{n-1}$ of the rays of this n -cone. \square

This suggests in functions f with finite interval valued directional derivatives $C^{\frac{1}{2}\ell}(\mathbb{R}^n) \supset C^{\frac{7}{8}\ell}(\mathbb{R}^n)$, the subset of all those with all $f'_u(x)$ forward continuous, i.e., from each x all directions can be allocated to finitely many closed n -cones $\phi^n(x)$ such that $\lim f'_u(y) = f'_u(x)$ for u belonging to, and y approaching x always staying in, one of these cones: we'll show this implies all tangent cones are unbroken, i.e. that indeed $C^{\frac{3}{4}\ell}(\mathbb{R}^n) \supset C^{\frac{7}{8}\ell}(\mathbb{R}^n)$:-

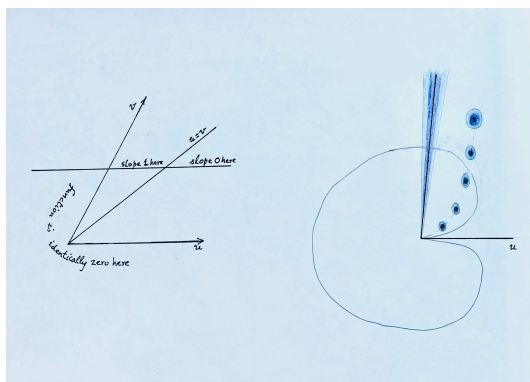
There's nothing to do for $n = 1$ and for $n > 2$ it suffices to prove continuity with respect to u of $f'_u(x)$ for restrictions of f to 2-planes through x . For $n = 2$ by bisecting cones if need be we can assume each closed cone $\phi^2(x)$ at x has an angle less than 180 degrees, so any vector v in it has both coordinates non negative with respect to the unit vectors u_1, u_2 along its bounding rays, that we take as our two axes: so we know that the finite interval valued $f'_{u_1}(y)$ and $f'_{u_2}(y)$, the partial derivatives parallel to axes, are continuous near the origin x

on the closed first quadrant. This already ensures the chain rule holds to the extent that the forward derivative at x along any $v = c_1u_1 + c_2u_2$ in $\phi^2(x)$ is given by $f'_v(x) = c_1f'_{u_1}(x) + c_2f'_{u_2}(x)$. Since there are only finitely many cones concatenation shows each $T_x f$ is even piecewise linear (but $C^{\frac{7}{8}}(\mathbb{R}^n)$ is bigger than $C^{PL}(\mathbb{R}^n)$, e.g., contains simplex-wise smooth functions). \square

These conditions are restrictive: notably, for single-valued functions $C^0(\mathbb{R}^n)$ denjoy-young-saks kicks in, so there is a unique $f'_u(x)$ a.e. on any directed line parallel to u , so even the condition $\lim f'_u(y) \in f'_u(x)$ as y reverses back to x on these rays implies single-valued finite forward derivatives. \square Only single-valued functions $C^0(\mathbb{R}^n) \cap C^{\text{any}}(\mathbb{R}^n)$ intervene in the definition of any structure, so this point also settles a problem raised before: a manifold admits a $C^{1\ell}$ -structure iff it admits a smooth structure:- denjoy-young-saks shows if $f \in C^0(\mathbb{R}^n) \cap C^{1\ell}(\mathbb{R}^n)$, then each $f'_u(x)$ is necessarily single-valued and continuous for all x , so this subring is no bigger than $C^1(\mathbb{R}^n)$, et cetera. \square Those ‘fat’ tangent cones with upto 2^n facets were of $C^{1\ell}$ interval-valued sections of $M^n \times \mathbb{R}$ or more generally of any of the $|H^1(M^n; \mathbb{Z}/2\mathbb{Z})|$ unoriented real line bundles on manifold M^n (or sections of the associated plane bundle with fibers $\mathbb{R} \oplus \mathbb{R}$ equipped with $(x, y) \leftrightarrow (y, x)$). In this sense we might say there are exactly $|H^1(M^n; \mathbb{Z}/2\mathbb{Z})|$ many $C^{1\ell}$ structures subordinate to each smooth structure on M^n and likewise for any other structure on a topological manifold. \square

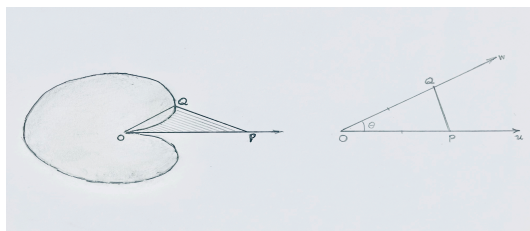
That ‘finitely many cones ...’ add-on, which made each $T_x(f)$ even piecewise linear, seems excessive; just $f \in C^{\frac{3}{4}}(\mathbb{R}^n)$, i.e., each $T_x f$ unbroken, i.e., angular continuity of forward derivatives in polar coordinates at x , seems right. However, this continuity in $n - 1$ angles notwithstanding, given any direction v other than u there are functions even in $f \in C^{PL}(\mathbb{R}^n)$ such that the limit of $f'_u(y)$ as y approaches x from this direction is not $f'_u(x)$; on the other hand, there are functions f not even in $C^{\frac{3}{4}}(\mathbb{R}^n)$ such that $f'_u(y)$ is continuous at x along v , and likewise for other preassigned transverse approaches to x :-

We can take $n = 2$ and $x = (0, 0)$ in coordinates parallel to u, v . Consider the piecewise linear function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ identically zero outside this quadrant, which in it on any line parallel to u has slope 1 till the line $u = v$, after which slope becomes 0: then $f'_u(0, 0) = 0$ but $f'_u(0, v) = 1$ for all $v > 0$. Being piecewise linear its tangent cones are unbroken, e.g., $T_{(0,0)}(f) = \text{graph}(f)$ since it is conical. On the other hand consider now that ‘heart example’ $\mathbb{R}^2 \rightarrow \mathbb{R}$ with tangent cone at $(0, 0)$ broken along u ; and smoothly modify this function over that thin blue cone with axis v , so that over the thinner dark blue cone this new function f is identically zero: then $f'_u(0, v) = 0$ for all $v \geq 0$ but brokenness of cone persists. Likewise given a sequence (u_i, v_i) lying say on a parabola, which converges to $(0, 0)$, we can smoothly modify the heart function over those small disks around it, so that for the new function $f'_u(u_i, v_i) = 0$ for all i but the tangent cone at the origin stays broken.



As for $f \in C^{PL}(\mathbb{R}^n)$, the full tangent cones $T_y(f)$ of an $f \in C^{\frac{3}{4}}(\mathbb{R}^n)$ cannot vary continuously with y near a non-linear $T_x(f)$, i.e., not an $(n-1)$ -plane:- Such continuity in a neighbourhood U of x implies f is Lipschitz there, so by Rademacher's theorem we would have $T_y(f)$ linear a.e. in U ; but it also implies $T_y(f)$ is non-linear like $T_x(f)$ near x . \square However our examples do not rule out continuity of directional derivatives of any $f \in C^{\frac{3}{4}}(\mathbb{R}^n)$ along transverse paths depending on f , and neither does Rademacher shut the door on utility of say subset of $C^{\frac{3}{4}}(\mathbb{R}^n)$ defined by $f'_u(x)$ continuous in direction u , i.e., unlike $C^{\frac{7}{8}}(\mathbb{R}^n)$ now glue can be on just one ray; but this seems insufficient for boundedness of oscillation of $T_x(f)$ to spread out, so let's say tentatively $C^{\frac{13}{16}}(\mathbb{R}^n)$ will be defined by adding to this also a mild boundedness condition that automatically ensures $C^{\frac{13}{16}}(\mathbb{R}^n) \subset C^{\frac{3}{4}}(\mathbb{R}^n)$ and try to fix this condition:- later?

Our heart function $f \in C^{\frac{1}{2}}(\mathbb{R}^2)$ is not lipschitz, nor quasiconformal, but its directional derivatives are all semi-continuous:- from $P \neq O$ on tangent ray at cusp till a nearby $Q \neq O$ on heart slope $\frac{f(Q)-f(P)}{PQ} = \frac{OQ}{PQ} = \frac{\sin \angle OPQ}{\sin \angle POQ} \neq 0$; if PQ remains parallel as it shrinks to O numerator remains constant and denominator goes to 0, so slope goes to ∞ ; ⁶ but on any line, f is piecewise smooth, so has a unique v -continuous $f'_v(x)$ in any direction v at any x . \square



Indeed, if f is lipschitz all $T_x(f)$ are unbroken:- Slope of f on $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$ is $\frac{f(Q)-f(O)}{OQ} = \frac{OP}{OQ} \frac{f(P)-f(O)}{OP} + \frac{PQ}{OQ} \frac{f(Q)-f(P)}{PQ}$. If $OP = OQ$ and $\theta = \angle POQ$ is so small that $\frac{PQ}{OQ} = 2 \sin \frac{\theta}{2} < \frac{\epsilon}{L}$, where L is the lipschitz bound on all slopes

⁶Otoh directional derivative at all such P along PO is zero, so f is not in $C^Q(\mathbb{R}^2)$, and its rotations $f : \mathbb{R}^n \rightarrow \mathbb{R}, n > 2$ not even in $C^{Q_{n-1}}(\mathbb{R}^n)$.

(in neighbourhood of O within which we are working), the second term is in any preassigned $(-\epsilon, \epsilon)$. Then, by shrinking $OP = OQ$ we can ensure as well that slopes on \overrightarrow{OP} and \overrightarrow{OQ} are within ϵ of the directional derivatives $f'_u(O)$ and $f'_w(O)$ in these directions. So f'_w is within 3ϵ of f'_u , and hence $f'_w \rightarrow f'_u$ as $\theta \rightarrow 0$. \square *Note directional derivatives above of such f are necessarily finite but can be interval-valued, these unbroken cones $T_x(f)$ are in general $(n + 1)$ -dimensional.* \square A natural question now arises: does this nice theorem ⁷ hold for quasiconformal functions $f \in C^Q(\mathbb{R}^n)$ or even if $f \in C^{Q_i}(\mathbb{R}^n), i < n$?

The argument above for the lipschitz $L = Q_0$ case suggests that at least an analogue ought to be true for any Q_i . A function changes the *weight* of a point (if we allow weights to be positive, zero or negative) that is it's a *0-measure* on manifold locally n -space (taking log we can keep positivity but let's not do that). The slope over a directed segment is its average rate of change, obviously it is single-valued, and directional derivative-postulated finite but not necessarily single valued-its instantaneous change at initial point in this direction as this segment shrinks to it, and tangent cone of function at this point visualizes these in all directions. We can do the same for *i-measures* again in same signed sense which are available at least when orientation is prescribed on n -space as *i-forms* (again note coefficient zero measure is average). One can now talk *over directed segments slopes of i-form* and taking limits or shrinking segments to initial point *instantaneous directional rates of change* of *i-measures*. The former are single-valued, the latter we again postulate finite but not necessarily single-valued. Their values now are forms, but applied to each *i*-subspace, thought of as an *i*-vector using an oriented orthonormal basis and *evaluating, i.e., sort of integrating* on it one again gets a same kind of pictured tangent cone on it. Indeed similar evaluation of the *i*-form on this *i*-vector gives us a function and one would think one just recovered the tangent cone in this particular case. Our notion of *i-quasiconformality* Q_i (so $i = 0$ is lipschitz, $i = 1$ quasiconformal) *seems very much to correspond to saying that slopes are bounded locally of this measure*, so the same argument as above, interpreting now L as this bound shows unbrokenness of the possibly different tangent cone defined over each *i*-space, and maybe for special *i*-forms equivalent to functions all over n -space. On *manifolds of class* Q_i by definition such functions are globally meaningful, and this unbrokenness is there for each function of this class, evaluated in turn on a fixed *bundle of i-measures* globally defined generalizing the familiar bundles of tangent *i*-covectors for the case of smooth manifolds. In particular for the 1-quasiconformal case we are saying that though possibly thicker and all kinky this generalized bundle of tangent covectors is unbroken at each point, also the local bound available tells us it can't jump around too much as we move to a nearby point. Therein we propose is the direct topological reason for that primal four dimensional obstruction to the existence of such a structure, and that it can be pin-pointed exactly and directly in the homotopy group considerations of Pontryagin and Rokhlin without going in the long long but no doubt super interesting detour via instantons and gauge fields etc through which it was

⁷(which precludes discontinuity of kind [?], p.16 for f lipschitz)