.....(last line of p.4).... Thus V(,F) is a functor — cf. § 4 of [6] — by means of which we are "visualizing" the category of incidences of the simplicial complex Δ.

One can play the zeta game with other algebraic geometric visualizations of Δ also. For example one has the **affine variety** covering $V(\Delta,F)$, i.e. the set $\hat{V}(\Delta,F)$ of points of $F^n \setminus \{0\}$ whose support belongs to Δ; so $V(\Delta,F)$ is the quotient of $\hat{V}(\Delta,F)$ under the action of the multiplicative subgroup $F^\times$ of the field F. The definition and computation of the zeta function of $\hat{V}$ is analogous to that of V.

Theorem ($\hat{2.2}$). $\hat{Z}_\Delta(t) = \prod_{j=1}^{d+1} \dfrac{1}{(1-q^j t)^{\hat{F}_{j-1}}}$, *where*

$(\hat{1})$ $$\hat{F}_{j-1} = F_{j-1} - F_j = \Sigma_{i=j-1}^{d} (-1)^{i+1-j} \cdot \binom{i+1}{j} \cdot f_i \text{ , and so}$$

$(\hat{2})$ $$f_j = \Sigma_{i=j-1}^{d} \binom{i+1}{j} \hat{F}_i .$$

This time the singular homology of the variety over $\mathbb{C}$ is given by the following (where $\sigma = \phi$ is allowed and reduced homology is used).

Theorem ($\hat{2.3}$). $\bar{H}_i(\hat{V}(\Delta,\mathbb{C});\mathbb{Z}) \cong \oplus_{\sigma\in\Delta} \bar{H}_{i-2|\sigma|}(lk_\Delta(\sigma);\mathbb{Z})$.

In fact $\hat{V}(\Delta,\mathbb{C})$, or equivalently its intersection $Sph_{\mathbb{C}}(\Delta)$ with the unit sphere of $\mathbb{C}^n$, has the *homotopy type* of the bouquet $\bigvee_{\sigma\in\Delta} (lk_\Delta(\sigma)).S^{2|\sigma|-1}$: cf. Ziegler-Zivaljevic's [11] who prove the analogous formula $\bar{H}_i(\hat{V}(\Delta,\mathbb{R});\mathbb{Z}) \cong \oplus_{\sigma\in\Delta} \bar{H}_{i-|\sigma|}(lk_\Delta(\sigma);\mathbb{Z})$ for the variety over $\mathbb{R}$.

Lemma (2.1). $\hat{F}_{j-1} = \Sigma_{|\sigma|=j}\ \bar{\chi}(lk_{\Delta}(\sigma))$.

The above formula (where reduced Euler characteristics of links are used) follows easily from $(\hat{1})$ and gives an analogous connection between the zeta function of $\hat{V}$ and its singular homology.

Following [6] it is useful also to think of $Sph_{\mathbb{C}}(\Delta) \simeq \hat{V}(\Delta,\mathbb{C})$ as **a small deleted join** of Δ with respect to the group $G = S^1 \simeq \mathbb{C}^{\times}$. For example see [7] which gives an interesting geometric application of the Chern class of the circle bundle $Sph_{\mathbb{C}}(\Delta) \longrightarrow Proj_{\mathbb{C}}(\Delta)$ $(= V(\Delta,\mathbb{C}))$; see also (3.9) below. By using higher linear groups $GL(m,F)$ one can also consider zetas of other affine and projective visualizations of Δ.

(after end of §3 on p. 7)

As was observed in [1], and later independently in [6], the formula $\bar{H}_i(\hat{V}(\Delta,\mathbb{R});\mathbb{Z}) \cong \oplus_{\sigma\in\Delta} \bar{H}_{i-|\sigma|}(lk_\Delta(\sigma);\mathbb{Z})$ has the striking corollary that Δ *is Cohen-Macaulay of dimension* d *iff the homology of* $\hat{V}(\Delta,\mathbb{R})$ *is trivial in dimensions less than* d. There is an analogous result for the complexification $\hat{V}(\Delta,\mathbb{C})$ of the real variety $\hat{V}(\Delta,\mathbb{R})$.

Theorem (3.$\hat{9}$). Δ *is Cohen-Macaulay of dimension* d *iff* $\dim(\bar{H}_i(\hat{V}(\Delta,\mathbb{C});\mathbb{C})) = (-1)^{i+1}.\hat{F}_{i-d-1}$ *for all* i (so it is zero in dimensions i < d).

Proof. If Δ is Cohen-Macaulay (2.$\hat{3}$) gives $\dim(\bar{H}_i(\hat{V}(\Delta,\mathbb{C});\mathbb{C})) = \Sigma_{|\sigma|\geq i-d}\ \bar{\beta}_{i-2|\sigma|}(lk_\Delta\sigma) = \Sigma_{|\sigma|=i-d}\ \bar{\beta}_{2d-i}(lk_\Delta\sigma) = \Sigma_{|\sigma|=i-d}(-1)^{i+1}.\bar{\chi}(lk_\Delta\sigma) = (-1)^{i+1}.\hat{F}_{i-d-1}$.

Conversely, we'll check by induction that for all σ with $|\sigma| \geq$ i-d one has $\bar{\beta}_j(lk_\Delta\sigma) = \emptyset$ for $j < i-2|\sigma|$. This suffices because for $|\sigma| \triangleq$ i-d it reads $\bar{\beta}_j(lk_\Delta\sigma) = \emptyset$ for $j < d-|\sigma|$. Thus the same calculation as above again gives $\Sigma_{|\sigma|=i-d}\ \bar{\beta}_{2d-i}(lk_\Delta\sigma) = (-1)^{i+1}.\hat{F}_{i-d-1}$. So $\dim(\bar{H}_i(\hat{V}(\Delta,\mathbb{C});\mathbb{C})) \leq (-1)^{i+1}.\hat{F}_{i-d-1}$ (or à fortiori the given equality) implies that we must have $\bar{\beta}_{i-2|\sigma|}(lk_\Delta\sigma) = \emptyset$ for all σ with $|\sigma| >$ i-d. Thus for all σ with $|\sigma| \geq$ i+1-d we have $\bar{\beta}_j(lk_\Delta\sigma) = \emptyset$ for $j < i+1-2|\sigma|$. *q.e.d.*

Similar characterizations of Cohen-Macaulayness can be made using small deleted joins over any given non-trivial group G (rather than just G = $\mathbb{R}^\times \simeq \mathbb{Z}/2$ or $\mathbb{C}^\times \simeq S^1$). To end this section we turn now to a characterization of Cohen-Macaulayness via the quotient variety.

Theorem (3.9). *Δ is Cohen-Macaulay of dimension* d *iff the Betti numbers of* $V(\Delta,\mathbb{C})$ *are as given in* (3.3) *and* (3.4).

Proof. To check sufficiency we will use the Gysin exact sequence of the circle bundle $\hat{V} \simeq \mathrm{Sph}_{\mathbb{C}}(\Delta) \longrightarrow \mathrm{Proj}_{\mathbb{C}}(\Delta) = V$ which runs

$$\rightarrow H^{i-2}(V) \rightarrow H^{i}(V) \rightarrow H^{i}(\hat{V}) \rightarrow H^{i-1}(V) \rightarrow \quad .$$

Here $H^{i-2}(V) \rightarrow H^{i}(V)$ denotes cup product with the Chern class of the circle bundle. For $V = \mathbb{C}P^{d}$, and so for any d-dimensional Δ, this map is nonzero for all even $2 \le i \le 2d$. So we must have $\dim(\bar{H}_i(\hat{V}(\Delta,\mathbb{C});\mathbb{C})) \le \beta^{\mathbb{C}}_{i-1} + \beta^{\mathbb{C}}_{i} - 1 = (-1)^{i+1}(F_{i-d-1} - F_{i-d}) = (-1)^{i+1}.\hat{F}_{i-d-1}$. So as in proof of $(\hat{3.9})$ one has equality and Δ is Cohen-Macaulay. *q.e.d.*

Thus for a Cohen-Macaulay Δ the above maps $H^{i-2}(V) \rightarrow H^{i}(V)$ behave exactly like for $\mathbb{C}P^{d}$: they are zero for i odd and have rank 1 for all even $2 \le i \le 2d$. We remark that (3.9) is of interest for f-vector theory because the second author has checked that the numbers $\beta^{\mathbb{C}}_{i}$ are preserved under Kalai's algebraic shifting.

(after end of §4 on p. 8)

(4.3) The usual functional equation between the values at t and t^{-1} rarely holds for the zeta function $Z_\Delta(q,t)$ of the singular varieties $V(\Delta,F)$. However there is often a "*functional equation*" of sorts between its values at q and 1-q. For example for any simplicial manifold Δ with zero Euler characteristic one has

$$(\hat{Z}_\Delta)'(q,\emptyset) = (-1)^{d+1}(\hat{Z}_\Delta)'(1-q,\emptyset), \tag{9}$$

where $(\hat{Z}_\Delta)'(q,t)$ denotes the derivative of $\hat{Z}_\Delta(q,t)$ with respect to t. Indeed using $(\hat{2}.2)$ we get $\hat{Z}_\Delta(q,t) = 1 + \hat{F}(q).t + o(t^2)$ where $\hat{F}(z) = \hat{F}_\emptyset z + \hat{F}_1.z^2 + \hat{F}_2.z^3 + \ldots$, and by $(\hat{2})$ $-\hat{F}(1-z) = f(z)$, where $f(z) = f_\emptyset z - f_1.z + f_2.z^2 - \ldots$. So (9) is equivalent to $f(1-q) = (-1)^{d+1}.f(q)$, i.e. the Dehn-Sommerville equations (8). See also [5] for remarks regarding the corresponding "*Riemann hypothesis*".

References contd.

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