How I learnt some well-known folklore

by

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This is a covering note for *On equivelar maps*: the first paragraph below describes how this paper—or rather fragment—came to be written, the second why it is frozen in mid-stride, and the third what came out of it. The first shows again why genuine teachers who do no research are as common as unicorns, the second that there is a sweeping generalization of platonic solids which should be more well-known than unicorns but isn't, and the third informs you that a full account of this beautiful theorem in everyday language is now available for the enjoyment of any bright high-school student.

(1) The five platonic solids, the crowning glory of Euclid's *Elements*, are well-known to all mathematicians, as is the fact that the regular maps {p,q} that can live on a closed surface which is topologically equivalent to a round sphere are these five only, i.e., {3,3}, {3,4}, {4,3}, {3,5} and {5,3}. By October 10, 2007, the date on which I had finished *Cacti and mathematics* – in which paper, using patterns on plants as my cue, I have given a popular exposition of the result just mentioned – I was wondering as to what happens if one removes this topological constraint**: does there exist a finite regular map {p,q} for any pairs of numbers p ≥ 3, q ≥ 3?** Surely, I reasoned, this natural question must have occurred to many since at least the time of Riemann, and its answer known by now, and if, per chance, such a sweeping generalization of the five platonic solids was true, it must surely be prominently mentioned in any of the three books devoted to such-like things that I happened to have in my library, viz., Coxeter's *Regular Polytopes*, Coxeter and Moser's *Generators and Relations for Discrete Groups*, and McMullen and Schulte's relatively recent, *Abstract Regular Polytopes.* However a quick and cursory browsing of these three books revealed nothing of this nature. This apparent absence, especially from the second book, which is mostly about describing symmetry groups of finite regular maps, coupled with the fact that at the end of their initial recap of regular maps in the first chapter of the third book, the authors point out that dropping the requirement of symmetry gives many more maps, which they call equivelar – these were called uniform tilings in my exposition – led me to believe that there might be some group theoretic obstruction to building all regular {p,q}'s, but probably one did have all equivelar {p,q}'s? So I limited myself to uniform tilings only, and soon had so many diverse constructions of the same in my bag that they seemed sufficient, but after typing up some I realized that I was still falling short … anyway, what I had already typed up seemed interesting enough, and that is what you'll find in *On equivelar maps* … but you'll note that its concluding section is frozen in mid-stride in a "pre"-version of sorts … here you'll see (p,q) instead of {p,q}, and the numbering of references is different … this is so because events had taken the following sudden turn at this point.

(2) For the planned bibliographical remarks in this section, I had decided to look at as many of the pertinent items listed in the extensive *Bibliography* of McMullen and

Schulte as was feasible, and the title of [437], a 1997 paper of H.J. Voss, had soon caught my fancy. This particular paper I still have not seen, but I was readily able to obtain from the web a related paper by the same author, viz., *Sachs triangulations and infinite sequences of regular maps on given type*, Disc. Math. 191 (1998), pp. 223-240, which has the following on its very first page: **"In 1974, Grünbaum [5] conjectured that a finite regular map of type {p,q} exists for each p ≥ 3 and q ≥ 3. In 1983, Vince [10] proved a generalization of Grünbaum's conjecture using Mal'cev's theorem about groups of matrices."** That was that—I thought—and turned my attention towards understanding A. Vince, *Regular combinatorial maps*, J. Combin. Theory B 35 (1983), pp. 256-277. It turns out that the result in question is an almost immediate corollary of this 1940 theorem of Malcev about finitely generated linear groups, and since everyone and his uncle knows that the symmetry groups of the infinite hyperbolic tilings are of this type, it seems fair to say that, **Grünbaum was conjecturing something that was known to be true for decades**, and which is so sweeping that it deserved to be **well-known!** That is, in the reasonable—not the mathematicians' rarefied—sense of this hackneyed term : but as my own example shows, it was not known to me even in 2007, and I have a feeling it is not known to the majority of mathematicians even today in 2010. I remark that Malcev's theorem – which, by the way, is also quite simple to prove – in fact gives us *infinitely many* finite regular maps for each non-platonic {p,q}, and what was really new in Vince's paper was that he notices that it applies also to the symmetry groups of the elegant combinatorial generalization of hyberbolic tilings that he was considering. I had noticed then that Vince's paper was listed as [436] in McMullen and Schulte's *Bibliography*, and some further browsing within this book done now has shown me that these authors make a parallel use of Malcev's theorem for their own, and equally elegant combinatorial generalization of hyperbolic tilings. Moreover, this section 4C of their book begins: "*It is folklore in Riemann surface theory that each regular tessellation* {p,q} *of the euclidean or hyperbolic plane is the universal* 2*-covering for an infinite number of finite regular maps of the same type* {p,q} *on closed surfaces."* So the result that I had been looking for was there in this book, only it was not where – because of its striking and basic nature – I had naively presumed that it would be, viz., in their initial recap of regular maps itself. In the sentence that was just quoted we have an example of another usage common chez mathematicians: **folklore!** A catch-all for key facts or insights of uncertain provenance which are common knowledge to some insiders, and so remain unpublished *as such* because not many brownie points can, alas, be earned by their exposition; however, from the point of view of understanding, it is definitely much easier for an outsider if somehow this oral tradition can be grasped *first*, complicated-looking things have then a way of becoming very simple indeed.

(3) Anyway, I have now done my bit to make this folklore more well-known : you can read from *"213, 16A" and mathematics*, which discusses in everyday language the mathematics underlying some architectural motifs, **an account, understandable to any bright high-school student, showing from scratch why one has a finite regular map {p,q} for any p ≥ 3 and q ≥ 3!** This is in the eighth and final 'lecture' of this paper, entitled "*Magic Carpet,"* and represents a known alternative route to this result using permutations instead of matrices, my own inspirations being Poincaré, and a short 108-year old paper of G. A. Miller, *Groups defined by the orders of two generators and the* *order of their product,* Amer. J. Math. 24 (1902), pp. 96-100. If one has an infinite regular tiling by p-gons, q at each vertex, then a group generated by two elements of order p and q with product of order 2—and no other relations—is infinite, which is hailed on page 54 of Coxeter and Moser thus: *"This is surely one of the most remarkable contributions of geometry to algebra. For, the algebraic proof* (MILLER 1902) *is excessively complicated, requiring separate considerations of many different cases."* This is a tad unfair, for, the finite groups of this type that Miller makes in droves by allowing more relations are—interpreted geometrically—nothing but finite regular maps {p,q}: one might even say that algebra has made a remarkable contribution to geometry! Miller's paper begins by an apparent emphasis on the fact that it is an *infinity* of these groups which is going to be exhibited for each non-platonic {p,q}, which to me suggests, though he gives no earlier reference, that the existence of at least one such group for each $p \geq 3$ and $q \geq 3$ was already known (maybe to even Cayley)? Much more is of course known now in 2010 about regular maps and their groups, but their full classification is hopelessly difficult, indeed, *any non-abelian finite simple group can be realized as the group of orientation-preserving symmetries of a finite regular map!* However, since this discussion shall be continued in the planned extensive Notes to *"213, 16A"* which I'll be posting on this website in due time, I'll conclude this covering note by pointing out that the photograph of Uccello's inlay work that you'll find in this paper on equivelar maps is there again in this newer paper, but now I prefer to call it a 'dodecahedragram' – my own coinage – which is shorter, and possibly more mellifluous, than the 'small stellated dodecahedron' used in Coxeter's *Regular Polytopes.*

P.S. I'd like to emphasize, for the sake of its less-experienced readers, that, despite appearances, *"213, 16A" and mathematics* is not just *about* mathematics, it *is* a mathematics paper. Many other results too, besides the one mentioned above, are proved in its 37 pages. To fully absorb these arguments, and so 'see' their great beauty, you'll need to read most of the paper line-by-line, and there are long stretches where you'll need to hang on to every word, and, besides, a certain amount of re-reading and pondering is also entailed. There is no getting around all this: as I've mentioned in the paper itself, one needs to *do* mathematics to appreciate *done* mathematics!

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