spin, theta, etc.

The results obtained in writing b b & b, though at first sight quite different, surprisingly circled back in a very natural way to what we had been doing before, viz., hobson's choice, garlands, etc., which I had resumed as follows, when my laptop went hors de combat for some weeks.

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Just from the fact that Jordan's 1870 Traité predates Poincaré's discovery of automorphic functions in 1881 it is clear that his method for solving equations could not have used compact tiles $\{n, n\}$ with angle sum 2π that we have been focussing on in the above. Indeed, *Jordan's method of page 380 used only the non compact*—all vertices for the seed case are now on the circle itself on which the unknown roots are situated—*tiles with angle sum zero.*

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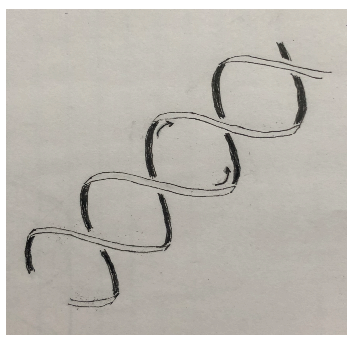
Now it seems better to postpone the above and detail that fun paper first. Apropos its last line, *all double helices below make a* 3*-sphere* :-



That is pairs $\pm q$ of right handed *helices* or directed *screws*—of slope one on cylinders of radius one with axes through origin—related by a half turn around a common axis. Since rotations act transitively and faithfully on all these helices they make an $SO(3) = \mathbb{R}P^3 = T^1(S^2)$, so pairs $\pm q$ of their 'square roots' make double covers of the unit tangent circles of S^2 whose union is S^3. Further, the half turn around the line through origin and the point on a helix at distance one reverses its direction, so all right handed screws make the base space of the double covering $T^1(S^2) \to T^1(\mathbb{R}P^2)$. $\square$ More explicitly,

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Our double helices are antipodal pairs of unit quaternions :- Each such pair $\pm q \in S^3$ gives by *conjugation* $q' \mapsto qq'q^{-1} = qq'\overline{q}$ exactly once all rotations of the S^2 of unit quaternions with *scalar part* zero. If ± 1 the identity, otherwise around the axis defined by the *vector part* of $\pm q$. For example since $e^{i\theta} i e^{-i\theta} = i$

and $e^{i\theta}je^{-i\theta} = j\cos 2\theta + k\sin 2\theta$ the quaternions $\pm e^{i\theta}$ are the two 'square roots' of the rotation which keeps $\pm i \in S^2$ fixed and rotates the perpendicular great circle by angle 2θ in direction j to k. Likewise conjugation with $e^{k\pi/4}$ keeps k fixed and rotates i to j, and conjugation with $q = e^{i\alpha}e^{k\beta}$ rotates *base point* i to any $v \in S^2$ having 'latitude' and 'longitude' $(2\alpha, 2\beta)$ with respect to $\pm i$ and $\pm k$ axes. Then conjugations by $\pm e^{i\theta}q$ will double cover the circle S^1_v of all rotations taking i to v. These *unit tangent circles* S^1_v are the disjoint fibers of the surjection p of all rotations on S^2 defined by $p(\rho) = \rho(i)$, in particular $S^1_{\pm v}$ are distinct copies of the great circle perpendicular to $\pm v$. For example conjugating with j takes i to $-i$ and keeps j fixed, so $\pm e^{i\theta}j$ are the square roots of the circle's worth of half turns S^1_{-i} which map i to $-i$ and keep $j\cos 2\theta + k\sin 2\theta$ fixed. Thus unlike rotations in S^1_i which restrict to rotations of the equator between $\pm i$, these half turns restrict to its reflections. Therefore the double covering disjoint circles $\tilde{S^1_v}$ into which S^3 partitions are the pre-images of the *Hopf map* $S^3 \to S^2$ defined by $q \mapsto qi\bar{q}$. $\square$

All above is on page 115 of Steenrod's *The Topology of Fibre Bundles* (1951) and reminded me–see PG&R IV, page 5–of *his 1969 letter:*

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Q: *On which regions $D \subset \mathbb{R}^3$ are all vector fields without zeros cross products?* Note cross product $v \times w$ is quaternion product vw plus dot product $v \cdot w$ and vector fields are continuous. **A:** *If and only if $H^2(D;\mathbb{Z}) = 0$*:-

Orthonormalizing we want regions on which any unit vector field $u(x)$ admits a pointwise perpendicular unit vector field $v(x)$, then there is a unique third $w(x)$ normal to both such that $u(x) = v(x) \times w(x)$. On S^2 the vector field $u(x) = x$ admits no such $v(x)$ for then we would have a homotopy of its *orientation reversing* antipodal map to identity along great semicircles from $-x$ to x through $v(x)$. In other words $p : T^1(S^2) \to S^2$ has no section. But, like any *fibration* p has the *homotopy lifting property*: for each homotopy u_t into the base there is another v_t into its total space such that $pv_t = h_t$. So all open sets $D \subset \mathbb{R}^3$ from which all maps into S^2 are trivial will do. Also, *Alexander duality* of 3-sphere $\widehat{\mathbb{R}^3}$ says $H^3(D;\mathbb{Z}) = 0$ always, so *obstruction theory* implies *degree* is a bijection of all homotopy classes of maps $D \xrightarrow{u} S^2$ with $H^2(D;\mathbb{Z})$. Hence all maps trivial iff $H^2(D;\mathbb{Z}) = 0$ *or dually iff $\widehat{\mathbb{R}^3} \setminus D$ is connected.* $\square$

Alas! I'm still looking for a copy of my full solution that I had sent him, but from his 1969 letters to me it amounted to the above, though I was ignorant or had only a muddled idea of almost all the italicized words, for example I had constructed the required covering homotopy from scratch.

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Musing on his 1969 letters I saw too *a smooth* **A:** *If and only if any divergence free vector field on D is a curl*:- $H^2(D;\mathbb{Z})$ being the reduced zeroth homology of $\widehat{\mathbb{R}^3} \setminus D$ is torsionfree, so it vanishes iff $H^2(D;\mathbb{R})$ vanishes, but $H^*(D;\mathbb{R})$ can be computed as kernels/images of the sequence

$$0 \to \mathcal{F}(D) \xrightarrow{\text{grad}} \mathcal{V}(D) \xrightarrow{\text{curl}} \mathcal{V}(D) \xrightarrow{\text{div}} \mathcal{F}(D) \to 0,$$

where $\mathcal{F}(D)$ and $\mathcal{V}(D)$ denote all smooth functions and vector fields on D, and the indicated $\mathbb{R}$-linear maps are the usual $\mathrm{grad}(f) = \nabla f$, $\mathrm{curl}(v) = \nabla \times v$ and $\mathrm{div}(v) = \nabla \cdot v$. The *composition of any two succesive maps is zero*, i.e., each image is a vector subspace of the kernel of the next map, and *de Rham's theorem* tells us that their graded quotient is $H^*(D;\mathbb{R})$. $\square$

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Steenrod's $\overrightarrow{P} = \overrightarrow{E} \times \overrightarrow{H}$ had at once reminded me of Poynting's *energy flux of an electromagnetic field* and so Maxwell. Helmholtz and Kelvin too had been drawn to the *finite dimensionality* of $H^*(D;\mathbb{R})$–for the domains they considered, more generally these *Betti numbers* are finite if D complements any *simplicial complex* K–but of course what these pioneers found fascinating is often entirely missing from textbooks on *fluid mechanics.*

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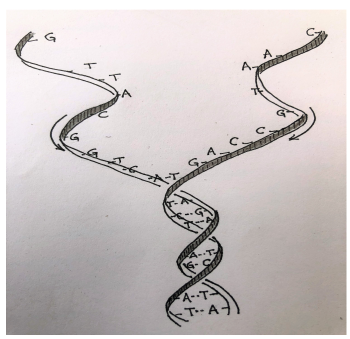
The right-handed screws in such texts are mnemonics, a left-hand rule can be given in each case. Naturally occuring helical braids in all life is a different thing altogether, and yes, some left-handed DNA is also now known, but it is rare: we who are left-handed are also molecularly right-handed! True, the backbones of the two DNA strands go from thick to thin in opposite directions, but the *direction that interests us is that in which genes unzip to reproduce*–by synthesis of the complementary bases on the single strands–two copies, this direction is necessarily the same for both strands :-

 $\square$

This pic is like Figure 11 of Watson's *The Double Helix* (1968) but before synthesis on its exposed strands. Note strands shown now are far from cylinderical, indeed DNA is very flexible and often "Amazing curves!" of incredibly long sequences of ordered base pairs are packed in tiny cells. However these sequences being finite this huge set of right-handed screws suggests *all right-handed arcs* which has (imho) the same homotopy type $T^1(\mathbb{R}P^2)$. A tightly-zippered double helix equipped with the direction in which it will unzip–this destiny if written in the sequence must be the opposite direction for the opposite sequence–is then a point ρ in the homotopy type $SO(3)$, and unzipping double covers it with the homotopy type S^3 of all exposed strands or square roots $\pm\sqrt{\rho}$.

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See Robin Kimmerer's Braiding Sweetgrass (2013) for more on the symbiosis between bamboo man and me. This important book I had received as a lock-down gift from my daughter Mallika Kaur, whose own recent book The Wheat Fields Still Whisper (2019) is equally important.

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The above homotopy type problem made explicit evokes $E = mc^2$ and more generally Frenet frames of arcs with Lorentz metrics in PG&R.

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Generalized cross is wedge product, usually of covectors; so *forms,* in above sequence euclidean metric of space is also involved. Undoubtedly much remains not yet understood in Hamilton's 18-part paper *On quaternions* (1844-50). He emphasized the (space-like) negativity of quaternion square of nonzero vectors. Dirac operators and so on have pushed this far but symplectization is much less understood than complexification.

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A coffee cup and a donut are the same to a topologist! This well known joke is usually attributed to Steenrod. From his only letter of 1970 to me–by then Princeton was within a round train trip–it seems it was one day of the first week of June 1970 that I had *the honour of being treated by Steenrod to a coffee with probably a donut!* I'm not a hundred percent sure because all was still new, I did not know what a donut was, but it was he who had ordered something nice and sweet with coffee for me in a coffee-shop on maybe Nassau Street. I recall that amongst many things we had talked about that day, he had been interested in my old effort to construct a meaningful *quaternionic analysis,* and had been happy to see my visual proof that $\pi_2(\ \ )$ *is abelian*: the same braided concretely in "213, 16A" now above the burbling stony brook. $\square$

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If the points s of a convex body $K^d \subset \mathbb{R}^d$ suffer any continuous displacement $v(s)$ a point $\hat{s}$ must stay put or be such that the body is in the half-space of directions making an obtuse angle with its displacement:- For, if the point $s+w(s)$ of K^d nearest to $s+v(s)$ is never s, segments $[s+w(s), s]$ produced show d-ball K^d homotopy equivalent to $(d-1)$-sphere ∂K^d. $\square$

As Milnor (1995) points out this result of Brouwer, applied to a product of n simplices with $v(s)$ the gradients in these n multidirections of n functions, gives at once the so-called Nash equilibria $\hat{s}$ of n-games. But the games of real interest in economics and other social sciences are in my (and if I read him right, Milnor's) opinion, ipso facto, too vague to benefit much from this result. Yes, some well-defined games, for example Hex, which Milnor also discusses, do have a beautiful theory. And, far more beautiful and deeper still are the justifiably celebrated but as such unrelated later results of Nash.

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From Nasar's *A Beautiful Mind* (1998) Steenrod was Nash's sounding board, informal adviser, and the first to grasp his beautiful result of 1952: any smooth closed manifold is diffeomorphic to a component of the zero-set of *polynomials!* So Nash was likely not one of the 'mathematicians whose methods of working I do not understand' in Steenrod's 1969 letter.

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Right-handed spin. The $SO(3)$ bundle over $\mathbb{R}^3$ of right-handed triads of unit vectors has because $\pi_1(\mathbb{R}^3) = 1$ a unique double covering $Spin(3) = S^3$ bundle of square roots. Their restrictions put orientation with spin on any object $P^3 \subset \mathbb{R}^3$ and if it is smooth compact and non-hollow induce, using *outward normal first* on its *surface* $\partial P^3 = M^2$ a circle bundle with a two-fold cover: the *'natural spin structure dictated by the double helices of life'* in b b & b. But obstruction theory shows, *if an oriented manifold has one, then it has exactly* $|H^1(\ ;\mathbb{F}_2)|$ *spin structures,* see Milnor (1963). So if this 'pretzel' P^3 has g holes it has 2^g right-handed spins—the others don't extend to $\mathbb{R}^3$—that induce on its oriented surface M^2 a square root of its 2^{2g} spin structures. $\Box$

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In analogy to what we used above if a not necessarily connected manifold has one it has exactly $|H^0(\ ;\mathbb{F}_2)|$ orientations; a 2-dimensional example having none is $\mathbb{R}P^2$, so it is not the boundary of any 3-manifold; but any 2-manifold or its disjoint union with $\mathbb{R}P^2$ occurs as such a boundary, etc., briefly: the *unoriented* 2*-cobordism group is* $\mathbb{Z}/2$*.* While, the *oriented* 2*-cobordism group is* 0 because surfaces occur as oriented boundaries of pretzels. Likewise, using now oriented manifolds equipped with a spin, one speaks of spin cobordism, for example, the *spin* 2*-cobordism group is* $\mathbb{Z}/2$:-

Let M^2 be oriented with a non-DNA or nonzero spin $\alpha \in H^1(M^2;\mathbb{F}^2)$. Choose a dual 1-cycle, perturb it in any pretzel P^3 bounded by M^2 to a simple closed curve and delete a thin 'donut' around it. This hollow object N^3 has as boundary the union of M^2 and a torus, so $H^1(N^3;\mathbb{F}_2)$ is 2-dimensional. Of its three nonzero spins one induces α on M^2 and a nonzero spin on the torus, and *'half cup square'* $H^1(M^2;\mathbb{F}^2) \xrightarrow{\vartheta} \mathbb{F}_2$ is nonzero on α iff this is the unique toral spin that is not a spin boundary, see Atiyah (1971). $\Box$

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This quadratic function ϑ was worked out in a manuscript left behind in 1866 by Riemann, and a problem on a par (and possibly tied) with his hypothesis on the zeta function is : *find* $\pi_*(S^2)$. Cobordism arose from it, e.g., $\pi_4(S^2) \cong \mathbb{Z}/2$ was seen as *a framed* 2*-cobordism group* :-

Equip the torus T^2 of unit quaternions $v = \frac{1}{\sqrt{2}}(e^{\mathbf{i}\alpha} + \mathbf{j}e^{\mathbf{i}\beta})$ with one of its two normal vector fields w tangent to S^3. The same torus with these normal frames (v, w) rotated by angles $\alpha + \beta$ will be $\mathcal{T}^2$. *No normally framed* 3*-manifold of* $\mathbb{R}^4 \times [0, 1]$ *has ends* T^2 *and* $\mathcal{T}^2$: see Pontryagin's *Smooth Manifolds and* *their Applications in Homotopy Theory* (1955), in particular pages 110-111. Surjecting suitably a tubular neighbourhood of the torus on an open ball B^2, and its complement on a point at infinity, gives a continuous map $S^4 = \widehat{\mathbb{R}^4} \to \widehat{B^2} = S^2$ having $\mathcal{T}^2$ as the pull-back of a regular value with a pair of orthonormal vectors; this map is homotopically non-trivial. $\square$

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Homotopically the above map is the Hopf fibration composed with its first suspension. Pontryagin's method dates back to 1938 when, using as obstruction to framed 1-cobordism the mod 2 linking due to $\pi_1(SO_{k+2}) \cong \mathbb{Z}/2$, he had shown that all its suspensions are non-trivial. So, thanks to $\pi_i(S^3) \cong \pi_i(S^2)$ for $i > 2$, which follows from the Hurewicz exact sequence of this fibration, $\pi_4(S^2) \cong \mathbb{Z}/2$ was already known. The merit of the above new look at the non-triviality of this group was by 1950 he had seen what he had missed in 1938, that framed 2-cobordism is obstructed by Riemann's characteristic ϑ, which moreover lives on in suspensions, so all these $\pi_{4+k}(S^{2+k})$ were in fact $\mathbb{Z}/2$, not 0 as he had claimed in 1938 because he overlooked the all-important term in this theta coming from the intersection form of the surface.

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The baton was passed on to Rokhlin who analyzed $\pi_5(S^2) \cong Z_2$ and its stem using framed 3-cobordism in four famous notes of 1951-52. Amusingly like his mentor he too initially slipped and concluded that a 3-torus $\mathcal{T}^3$, with its obvious normal framing twisted by the sum of its 3 angles, was stably the boundary of a certain smooth framed M^4, which he made using an L^4 with an 'honest' 2-sphere in it homologous to a certain 2-torus. After the fourth note, in which are made the corrections, it was clear that there are in his L^4 only singular 2-spheres of this kind: Rokhlin's L^4 gives counterexamples to the 2-dimensional Whitney trick. See Guillou and Marin's *À la Recherche de la Topologie Perdue* (1986), especially pages 5-8 and 38-42. The fourth note outlined why the oriented 3-cobordism group is zero, while signature of the intersection form is a homomorphism from the oriented 4-cobordism group to $\mathbb{Z}$. Using these tools he saw in tandem that the Freudenthal-stable groups $\pi_{5+k}(S^{2+k}), k \geq 3$ were $\mathbb{Z}/24$, and that, if all surfaces of a closed smooth oriented 4-manifold have even self-intersection, then its signature is divisible by 16. See Finashin and Kharlamov (2020) who analyze this part of Rokhlin's argument and show it also gives the other unstable groups of this stem: $\pi_6(S^3) = \mathbb{Z}/12, \pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}/12$.

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Talking of 'topologie perdue' van Kampen (1932) had before Stiefel a mod 2 cohomology class by counting under any general position map $K^n \to \mathbb{R}^{2n}$ for pairs of disjoint n-simplices whether their intersections were even or odd; and for $n \neq 2$ had sketched the sufficiency of its vanishing for embeddabilty. My talks and writings did much (imho) to give this classic due importance, in particular

I pointed out that the smooth Whitney trick for $n > 2$ which was then in the future was not really needed, simple conical constructions did the job; and as expected Rokhlin's L^4 provides simple examples to show insufficiency for $n = 2$; but left open is the question whether a homotopy theoretic necessary condition, which for $n > 2$ is equivalent, suffices for $n = 2$; for more musings on these things see Switches and Fingers.

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Since ± 1 is not in Clifford's torus $T^2 \subset \mathbb{S}^3$, in 2-planes parallel to this and a new axis replacing pairs of points of S^3 with circles having these diameters gives a 3-torus T^3 in S^4 the unit sphere of $\mathbb{R}^5$; then revolving again a $T^4 \subset S^5 \subset \mathbb{R}^6$, and so on. Choose one of its two unit normal vector fields $w(v), v \in T^{n-2}$ tangent to S^{n-1} and twist this normal framing (v, w) using any continuous function $f(\alpha_1 = \alpha, \alpha_2 = \beta, \alpha_3, ..., \alpha_{n-2})$ of angles to get a $\mathcal{T}^{n-2}$ and then by Pontryagin's method a map $S^n = \widehat{\mathbb{R}^n} \to \widehat{B^2} = S^2$ having $\mathcal{T}^{n-2}$ as the pull-back of a framed regular value. *Can we get with this toral construction for any $n > 2$ a non-trivial element of $\pi_n(S^2)$?* I don't know, but with just the sum function $\alpha_1 + \cdots + \alpha_{n-2}$ the answer is no: the map $S^n \to S^2$ we get with this twisting is homotopic to a composition of suspensions of the Hopf map, see Guillou-Marin, but the 4-stem starting with $\pi_{10}(S^6)$ is zero. $\square$

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We note the tribar in the impossible art of Escher et al, and its kin doodled in b b & b with other polygonal, even polytopal, sections are not only topologically possible, *'impossible' tribars make the Hopf map and its suspensions*: subdivide the pull-back of a regular cell in the base space into three prisms, were this cell structure realizable flatly–that is were these three constituent bars and the three ells all flat–in any euclidean space, there would be no twisting, contradicting the homotopic non-triviality. $\square$ We note this non-embeddability is stronger than if we were to subdivide the three prisms further into simplices, when certainly we can embed with all simplices flat in a euclidean space of high enough dimension. This here is akin to how Pontryagin's number of $\mathbb{C}P^2$ obstructs the cell-wise flat embeddability in any euclidean space of the Poincaré dual of any triangulation, but now we are talking of a similar phenomenon for some subdivisions of S^3 thus also of any 3-manifold, and even of some subdivisions of any non-orientable M^2 using the Möbius strip through which bing jumps in this picture story to return now to the front door of his ome.

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Rokhlin's results played a key rôle in the existence and classification of smooth and piecewise linear structures on closed manifolds. In PG&R we saw that all this cartesian matter is but different forms of motion, and if one gives up the unrealistic axiom that ambient space has infinite extent, that is one adopts the relativistic viewpoint, *manifolds inherit a Lipschitz structure from* *the birthing motion.* Conversely per Sullivan (1976) outside dimension four any closed manifold has a unique Lipschitz structure. So, impressive though the aforementioned classifications are, most important to me seems this unresolved *conjecture : S^4 has precisely two Lipschitz structures!*

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From Steenrod's second letter of 1969 it seems I had plonked first for the condition $\pi_2(D) = 0$ but quickly took this move back–unlike chess mathematics is much kinder this way!–maybe because I saw that a tubular neighbourhood D of any surface $M^2 \neq S^2$ has $H^2(D;\mathbb{Z}) \neq 0$ but $\pi_i(D) = 0\,\forall i \geq 2$:- the universal covering space of M^2 being contractible. $\square$

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In the manuscript left behind by Riemann in 1866 any M^2 was viewed as a graph $w = \sqrt{f(z)}$ where $f(z)$ is a polynomial with distinct roots to compute $\vartheta : H^1(M^2;\mathbb{F}_2) \to \mathbb{F}_2$. We'll instead view M^2 as the index 2 quotient of a half-turn tiling of an open disk, generated by a circularly curved polygon with angle sum 2π, to carry out this same calculation :-

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K S Sarkaria