Straight to Mecca, Notes 7-8.

7. The polar equation of C_k, the envelope of circles with diameters NP, $P \in C_{k-1}$, can be verified thus. Take N as the origin, x-axis along NM, the y-axis perpendicular to it, and denote the length NM by R. Then the given circles have centres $(a, b) = (\frac{1}{2}R\cos^k(\frac{\theta}{k})\cos\theta, \frac{1}{2}R\cos^k(\frac{\theta}{k})\sin\theta)$ and pass through the origin. From Goursat – see page 432 bottom – the *chord of contact* of each circle, i.e., the straight line determined by the two intersections of this circle with an infinitely close circle of the family, has equation $xa' + yb' = 0$, where primes denote differentiation with respect to the parameter θ. The envelope is the union of the loci of these two intersections, that is, the single point N, union the locus of the other intersection Q. The tan of the angle which NQ makes with the x-axis is $-a'/b'$. Computing these derivatives a' and b' – their common factors $\frac{1}{2}R\cos^{k-1}(\frac{\theta}{k})$ cancel out – we get $\tan(\theta + \frac{\theta}{k})$. So NQ makes an angle $\theta + \frac{\theta}{k}$ with the x-axis, and has length $R\cos^k(\frac{\theta}{k})\cos(\frac{\theta}{k}) = R\cos^{k+1}(\frac{\theta}{k})$. That is, Q is the point on $r(\phi) = R\cos^{k+1}(\frac{\phi}{k+1})$ for $\phi = \theta + \frac{\theta}{k}$.

8. The path WTM of Figure 1 in fact minimizes distance amongst all paths from W to M that don't go into latitudes more northerly than W. For, once we are on a tangent line through M to the latitude of W we should go on it to M. So our path is in the closed cone bounded by the tangent lines MT and MT'. So it must go through T or T', and it must stay on the latitude of W till this point, for we can shorten any other path of the stated kind by bypassing a southern-most point by a small line segment.