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THE ZETA FUNCTION OF A SIMPLICIAL COMPLEX

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ABSTRACT

Given a simplicial complex Δ on vertices $\{1,\dots,n\}$ and a field F we consider the subvariety of projective $(n-1)$-space over F consisting of points whose homogeneous coordinates have support in Δ. We give a simple rational expression for the zeta function of this singular projective variety over F_q and show a close connection with the Betti numbers of the corresponding variety over $\mathbb{C}$. This connection is particularly simple in the case when Δ is Cohen–Macaulay.

1. Introduction

Let Δ be a simplicial complex on vertex set $\{1,\dots,n\}$ and for any given field F let $V(\Delta, F)$ be the set of points in FP^{n-1} (projective $(n-1)$-space over F) whose support (set of nonzero coordinate positions) belongs to Δ. This is a projective variety, being the union of linear subspaces. By the **zeta function** of Δ over a

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finite field F_q we will mean the zeta function of this variety in the standard sense of algebraic geometry, namely

$$Z_\Delta(q,t) = \exp\left(\sum_{k\geq 1} \mathrm{card} V\left(\Delta, F_{q^k}\right)\frac{t^k}{k}\right).$$

In this paper we will compute a simple rational expression for $Z_\Delta(q,t)$, which via the work of Ziegler and Živaljević [12] shows that the zeta function has intimate connections with the singular homology of the complex projective variety $V(\Delta,\mathbb{C})$. In fact, the filtration of the space $\|\Delta\|$ by the coskeleta of Δ provides Betti numbers that determine both the rational function $Z_\Delta(q,t)$ and the Betti numbers of $V(\Delta,\mathbb{C})$.

One possible interest in these observations lies in the fact that the varieties $V(\Delta,F)$ are highly *singular*, so that available theory does not seem to be able to predict such a precise connection between counting over finite fields and topology over $\mathbb{C}$. As is well known, in the case of *nonsingular* projective varieties the Weil conjectures [2, 4] give very detailed information of this sort.

In the nonsingular case it follows from the Deligne–Grothendieck–Weil theorem that the Betti numbers of the complex variety are determined by the zeta function. This is not in general true for the simplicial complex varieties $V(\Delta,\mathbb{C})$ considered here, but we will show a tight connection in the important case of Cohen–Macaulay complexes.

Another possible interest in this work is for potential use in f-vector theory. As we remark in Section 4 all questions about f-vectors of Cohen–Macaulay complexes can be translated into questions about Betti numbers of the corresponding varieties. This is reminiscent of the situation for polytopes and toric varieties, see [9], although the varieties $V(\Delta,\mathbb{C})$ unfortunately lack many of the good special properties of toric varieties.

2. The basic results

We will assume familiarity with the basic combinatorial and topological properties of (abstract) simplicial complexes; see Munkres [5] for this material. Throughout the paper Δ will be a simplicial complex of dimension d on vertex set $\{1,\ldots,n\}$. The **f-vector** of Δ is $f(\Delta) = (f_0, f_1, \ldots, f_d)$, where f_j is the number of j-dimensional faces. Let

$$f_j^* := \sum_{i=j}^{d} (-1)^{i-j} \binom{i}{j} f_i, \quad \text{for } j = 0, \ldots, d, \tag{1}$$

and call $f^*(\Delta) = (f_0^*, f_1^*, \ldots, f_d^*)$ the **f^*-vector** of Δ. The relation (1) can be inverted,

$$f_j = \sum_{i=j}^{d} \binom{i}{j} f_i^*. \tag{2}$$

The **coskeleton** $\Delta^{\geq j}$ is defined by $\Delta^{\geq j} = \{\sigma \in \Delta : \dim \sigma \geq j\}$. We think of $\Delta^{\geq j}$ as a simplicial complex by passing to its "order complex", i.e., the faces of $\Delta^{\geq j}$ are the chains $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_s$ with $\sigma_i \in \Delta^{\geq j}$. From this point of view $\Delta^{\geq 0}$ is the barycentric subdivision of Δ, so the chain of inclusions $\Delta^{\geq d} \subset \Delta^{\geq (d-1)} \subset \cdots \subset \Delta^{\geq 0}$ gives a filtration of the space $\|\Delta\| \cong \|\Delta^{\geq 0}\|$.

Lemma 2.1: *$f_j^* = \chi(\Delta^{\geq j})$, for $j = 0, \ldots, d$.*

Proof: The computation is most easily done using the theory of Möbius functions, see [10, Chap. 3]. Let $P = \Delta^{\geq j} \cup \{\hat{0}, \hat{1}\}$, the poset of faces of Δ of dimension $\geq j$ augmented by a top and a bottom element, and let μ be its Möbius function. By Hall's theorem [10, p. 120] we have

$$\chi(\Delta^{\geq j}) = \mu(\hat{0}, \hat{1}) + 1. \tag{3}$$

If $\dim \sigma = i$ ($j \leq i \leq d$) then $P_{\leq \sigma} = \{\tau \in P | \tau \leq \sigma\}$ is a lower-truncated Boolean algebra (all subsets of size $\geq j+1$ in an $(i+1)$-set), so $\mu(\hat{0}, \sigma) = (-1)^{i-j-1}\binom{i}{j}$. Hence

$$\mu(\hat{0}, \hat{1}) = -\sum_{\sigma < \hat{1}} \mu(\hat{0}, \sigma) = -1 + \sum_{i=j}^{d} f_i (-1)^{i-j} \binom{i}{j},$$

which with (1) and (3) proves the result. ∎

The zeta function of Δ over F_q (the finite field of order q) was defined in the introduction. In the following we suppress the prime power q from the notation. It follows from the work of Dwork [3] that $Z_\Delta(t)$ is a rational function. In fact, it has the following explicit form.

Theorem 2.2:

$$Z_\Delta(t) = \prod_{j=0}^{d} \frac{1}{(1 - q^j t)^{f_j^*}}.$$

Proof: The points $x = (x_1, \ldots, x_n)$ on the projective variety $V(\Delta, F_{q^k})$ are partitioned by their supports $\{i : x_i \neq 0\} \in \Delta$. If $\dim \sigma = i$ then there are clearly $\left(q^k - 1\right)^i$ points with support equal to σ (each of the $i+1$ nonzero positions can be filled in $q^k - 1$ ways and then we must divide by $q^k - 1$ to account for projective equivalence). Hence,

$$\begin{aligned}
\sum_{k\geq 1} \operatorname{card} V(\Delta, F_{q^k}) \frac{t^k}{k} &= \sum_{k\geq 1} \sum_{i=0}^{d} f_i \left(q^k - 1\right)^i \frac{t^k}{k} \\
&= \sum_{k\geq 1} \sum_{i=0}^{d} \sum_{j=0}^{i} f_i (-1)^{i-j} \binom{i}{j} q^{kj} \frac{t^k}{k} \\
&= \sum_{j=0}^{d} \sum_{k\geq 1} \frac{\left(q^j t\right)^k}{k} \sum_{i=j}^{d} (-1)^{i-j} \binom{i}{j} f_i \\
&= \sum_{j=0}^{d} f_j^* \sum_{k\geq 1} \frac{\left(q^j t\right)^k}{k} \\
&= \sum_{j=0}^{d} f_j^* \log \frac{1}{1 - q^j t} \\
&= \log \left(\prod_{j=0}^{d} \frac{1}{(1 - q^j t)^{f_j^*}} \right). \qquad \blacksquare
\end{aligned}$$

The following formula for the singular homology of the complex projective variety $V(\Delta, \mathbb{C})$ is due to Ziegler and Živaljević [12, Proposition 2.15]. There is a misprint in the original formulation; for a correction and additional discussion see [11, Corollary 6.7].

THEOREM 2.3 (Ziegler and Živaljević):

$$H_i(V(\Delta, \mathbb{C}); \mathbb{Z}) \cong \bigoplus_{j=0}^{d} H_{i-2j}\left(\Delta^{\geq j}; \mathbb{Z}\right).$$

Thus, the Betti numbers of the coskeleta $\Delta^{\geq j}$ determine, on the one hand, the Betti numbers of $V(\Delta, \mathbb{C})$ and, on the other, the zeta function of Δ.

COROLLARY 2.4: $\chi(V(\Delta, \mathbb{C})) = f_0$.

Proof: With the obvious choice of notation (see definition (5) below) we have

$$\chi(V(\Delta,\mathbb{C})) = \sum_{i=0}^{2d}(-1)^i\beta_i^{\mathbb{C}} = \sum_{i=0}^{2d}\sum_{j=0}^{d}(-1)^i\beta_{i-2j}^{\geq j}$$
$$= \sum_{j=0}^{d}\chi\left(\Delta^{\geq j}\right) = \sum_{j=0}^{d} f_j^* = f_0.$$

For the last equality see (2). ∎

We will end this section with an informal description of what the varieties $V(\Delta, F)$ "look like". The discussion will use the complex field $F = \mathbb{C}$ for the relevant geometric visualization but it is in principle completely general.

As a first small example let us take for Δ the boundary of a 2-simplex, i.e. Δ has 3 vertices and 3 edges. Then $V(\Delta,\mathbb{C})$ is homeomorphic to the union of three touching billiard balls (surface only). The i-th ball is the set of projective points (z_1, z_2, z_3) with $z_i = 0$ $(i = 1, 2, 3)$, which is a copy of $\mathbb{C}P^1 \cong S^2$, and two balls touch in $z_i = z_j = 0$ which is $\mathbb{C}P^0$ = point. This description immediately generalizes to any 1-dimensional complex Δ: the edges of Δ become 2-spheres in $V(\Delta,\mathbb{C})$ and vertices become points of contact. One can visualize $V(\Delta,\mathbb{C})$ by thinking of the graph Δ and then replacing each edge by a thin tube pinched at the endpoints.

For a general simplicial complex and a general field the picture is entirely similar. Each j-dimensional face is in $V(\Delta, F)$ represented by a copy of FP^j (living on the corresponding coordinate positions), and these projective spaces are glued together by inclusions in the same pattern as that of Δ. Thus $V(\,\cdot\,, F)$ is a functor — cf. §4 of [7] — by means of which we are "visualizing" the category of incidences of the simplicial complex Δ.

One can play the zeta game with other algebraic geometric visualizations of Δ also. For example, one has the affine variety covering $V(\Delta, F)$, i.e. the set $\widehat{V}(\Delta, F)$ of points of $F^n \setminus \{0\}$ whose support belongs to Δ; so $V(\Delta, F)$ is the quotient of $\widehat{V}(\Delta, F)$ under the action of the multiplicative subgroup F^* of the field F. The definition and computation of the zeta function of $\widehat{V}$ is analogous to that of V.

THEOREM 2.5:

$$\widehat{Z}_\Delta(t) = \prod_{j=0}^{d+1}\frac{1}{(1-q^j t)^{\widehat{f}_j}},$$

where

$$\widehat{f}_j = f^*_{j-1} - f^*_j = \sum_{i=\max\{j-1,0\}}^{d} (-1)^{i+1-j} \binom{i+1}{j} f_i. \tag{4}$$

This time the singular homology of the variety over $\mathbb{C}$ is given by the following (where $\sigma = \emptyset$ is allowed and reduced homology is used).

THEOREM 2.6 ([12]):

$$\widetilde{H}_i\left(\widehat{V}(\Delta, \mathbb{C}); \mathbb{Z}\right) \cong \bigoplus_{\sigma \in \Delta} \widetilde{H}_{i-2|\sigma|}\left(\mathrm{lk}_\Delta(\sigma); \mathbb{Z}\right).$$

In fact $\widehat{V}(\Delta, \mathbb{C})$, or equivalently its intersection $\mathrm{Sph}_{\mathbb{C}}(\Delta)$ with the unit sphere of $\mathbb{C}^n$, has the homotopy type of the bouquet $\bigvee_{\sigma\in\Delta}\left(\mathrm{lk}_\Delta(\sigma) * S^{2|\sigma|-1}\right)$; cf. Ziegler–Živaljević [12] who prove the analogous formula

$$\widetilde{H}_i\left(\widehat{V}(\Delta, \mathbb{R}); \mathbb{Z}\right) \cong \bigoplus_{\sigma \in \Delta} \widetilde{H}_{i-|\sigma|}\left(\mathrm{lk}_\Delta(\sigma); \mathbb{Z}\right)$$

for the variety over $\mathbb{R}$.

LEMMA 2.7:

$$\widehat{f}_j = -\sum_{|\sigma|=j} \widetilde{\chi}\left(lk_\Delta(\sigma)\right), \qquad \textit{for } 1 \le j \le d+1,$$
$$\widehat{f}_0 = -\chi(\Delta).$$

These formulas (where reduced Euler characteristics of links are used in the first one) follow easily from (4) and give a connection between the zeta function of $\widehat{V}$ and its singular homology analogous to that for V.

Following [7] it is also useful to think of $\mathrm{Sph}_{\mathbb{C}}(\Delta) \simeq \widehat{V}(\Delta, \mathbb{C})$ as a small deleted join of Δ with respect to the group $G = S^1 \simeq \mathbb{C}^*$. For example, see [8], which gives an interesting geometric application of the Chern class of the circle bundle $\mathrm{Sph}_{\mathbb{C}}(\Delta) \to V(\Delta, \mathbb{C})$. By using higher linear groups $\mathrm{GL}(m, F)$ one can also consider zeta functions of other affine and projective visualizations of Δ.

For other aspects of the varieties associated to simplicial complexes, see [1, §11.2] and [7, §4].

3. The Cohen–Macaulay case

Let us now assume that Δ is a Cohen–Macaulay complex (in characteristic zero). Topologically this means that reduced homology with complex coefficients vanishes below the top dimension both for Δ itself and for links $\mathrm{lk}_\Delta(\sigma)$ of all faces $\sigma \in \Delta$. Algebraically it means that the Stanley–Reisner ring $\mathbb{C}[\Delta]$, or equivalently the homogeneous coordinate ring of $V(\Delta, \mathbb{C})$, is Cohen–Macaulay. Some examples of Cohen–Macaulay complexes are Tits buildings, matroid complexes and triangulations of spheres. See Stanley [9] for a thorough discussion of this notion and its background in commutative algebra.

The following is an immediate consequence of the "rank-selection theorem" [9, Theorem III.4.5] applied to the face lattice of Δ.

LEMMA 3.1: *If Δ is Cohen–Macaulay then so is every coskeleton $\Delta^{\geq j}$.*

Hence, because of the vanishing of reduced homology below the top dimension, we deduce from this and Lemma 2.1:

COROLLARY 3.2: *If Δ is Cohen–Macaulay of dimension d, then*

$$f_j^* = \begin{cases} 1 + (-1)^{d-j}\beta_{d-j}\left(\Delta^{\geq j}\right), & j = 0, \dots, d-1, \\ \beta_0\left(\Delta^{\geq d}\right), & j = d. \end{cases} \quad \blacksquare$$

From now on we assume that Δ is Cohen–Macaulay of dimension d, and we use the abbreviated notation

$$(5) \qquad \beta_i^{\mathbb{C}} := \dim_{\mathbb{C}} H_i(V(\Delta, \mathbb{C}); \mathbb{C}) \qquad \text{and} \qquad \beta_i^{\geq j} := \dim_{\mathbb{C}} H_i(\Delta^{\geq j}; \mathbb{C})$$

for the respective Betti numbers. The variety $V(\Delta, \mathbb{C})$ is $2d$-dimensional, so $\beta_i^{\mathbb{C}} = 0$ for $i > 2d$ and $i < 0$. For the remaining cases we get the following.

PROPOSITION 3.3: *For Δ Cohen–Macaulay we have*

$$0 \leq i < d \Longrightarrow \beta_i^{\mathbb{C}} = \begin{cases} 1, & i \text{ even}, \\ 0, & i \text{ odd}, \end{cases}$$

$$0 \leq j \leq d \Longrightarrow \beta_{2d-j}^{\mathbb{C}} = \begin{cases} \beta_j^{\geq d-j}, & j = 0 \text{ or } j \text{ odd}, \\ \beta_j^{\geq d-j} + 1, & j > 0 \text{ and } j \text{ even}. \end{cases}$$

Proof: This results from substituting

$$\beta_i^{\geq j} = \begin{cases} 0, & 0 < i < d-j \\ 1, & i = 0 \text{ and } j < d \end{cases}$$

into $\beta_i^{\mathbb{C}} = \sum_{j=0}^{d} \beta_{i-2j}^{\geq j}$ (Theorem 2.3). ∎

Putting this information together with Corollary 3.2 we obtain the following direct relationship between the f^*-vector of a Cohen–Macaulay complex and the essential Betti numbers $\beta_i^{\mathbb{C}}$ $(d \leq i \leq 2d)$ of its corresponding variety $V(\Delta, \mathbb{C})$.

PROPOSITION 3.4: *For Δ Cohen–Macaulay we have $(0 \leq j \leq d)$:*

$$f_{d-j}^* = \begin{cases} \beta_{2d-j}^{\mathbb{C}}, & j \text{ even}, \\ 1 - \beta_{2d-j}^{\mathbb{C}}, & j \text{ odd}. \end{cases}$$ ∎

We can now formulate the consequence of all this for the zeta function.

THEOREM 3.5: *If Δ is Cohen–Macaulay, then*

$$Z_\Delta(t) = \prod_{j=0}^{d} \left(1 - q^{d-j}t\right)^{(-1)^{j+1}\beta_{2d-j}^{\mathbb{C}} - \delta_j},$$

where $\delta_j = 1$ if j is odd and $= 0$ otherwise.

Proof: This is a direct consequence of Theorem 2.2 and Proposition 3.4. ∎

The formula

$$(6) \qquad Z_\Delta(t) = \frac{\left(1 - q^{d-1}t\right)^{\beta_{2d-1}^{\mathbb{C}} - 1} \left(1 - q^{d-3}t\right)^{\beta_{2d-3}^{\mathbb{C}} - 1} \cdots}{(1 - q^d t)^{\beta_{2d}^{\mathbb{C}}} \left(1 - q^{d-2}t\right)^{\beta_{2d-2}^{\mathbb{C}}} \cdots}$$

(i.e., Theorem 3.5) shows that in the Cohen–Macaulay case the zeta function of Δ and the Betti numbers of $V(\Delta, \mathbb{C})$ mutually determine each other. Let us illustrate this formula with a few examples.

Example 3.6: Let Δ be the full simplex of all subsets of $\{1, \ldots, n\}$. Then $f_j^* = 1$ for all $j = 0, \ldots, d = n - 1$, since $\Delta^{\geq j}$ is a cone (as a poset it has a maximal element). Hence (by Theorem 2.2)

$$Z_\Delta(t) = \prod_{j=0}^{d} \frac{1}{1 - q^j t}.$$

On the other hand, $V(\Delta, \mathbb{C}) = \mathbb{C}P^d$ (the full projective space), which has Betti numbers $\beta_i^{\mathbb{C}} = 1$ if i is even and $0 \leq i \leq 2d$, and $\beta_i^{\mathbb{C}} = 0$ otherwise. This produces via formula (4) the same expression for $Z_\Delta(t)$.

Example 3.7: Let Δ be a 1-dimensional Cohen–Macaulay complex (i.e., a connected graph) with v vertices and e edges. Then $f_0^* = \chi(\Delta) = v - e$ and $f_1^* = \chi\left(\Delta^{\geq 1}\right) = e$, so

$$Z_\Delta(t) = (1-t)^{e-v}(1-qt)^{-e}.$$

From the topological description of $V(\Delta, \mathbb{C})$ given at the end of Section 2 (inflate the edges of Δ to thin tubes) one computes by elementary considerations the Betti numbers $\beta_0^{\mathbb{C}} = 1$, $\beta_1^{\mathbb{C}} = 1 + e - v$, $\beta_2^{\mathbb{C}} = e$, which via (6) also yields $Z_\Delta(t)$.

Example 3.8: Let Δ be the boundary complex of a $(d+1)$-simplex. Then $\Delta^{\geq j}$ consists of all proper subsets of cardinality $\geq j+1$ in $\{1, \ldots, d+2\}$, so $f_j^* = \chi\left(\Delta^{\geq j}\right) = 1 + (-1)^{d-j}\binom{d+1}{j}$ for $j = 0, \ldots, d$. (Cf. the proof of Lemma 2.1.) In this case it is not as easy to compute the Betti numbers of $V(\Delta, \mathbb{C})$ by inspection, but via Proposition 3.3 and 3.4 we get

$$\beta_{2d-j}^{\mathbb{C}} = \begin{cases} 1 + \binom{d+1}{j+1}, & \text{if } 0 \leq j \leq d \text{ and } j \text{ even}, \\ \binom{d+1}{j+1}, & \text{if } 0 \leq j \leq d \text{ and } j \text{ odd}, \\ 1, & \text{if } d < j \leq 2d \text{ and } j \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

The variety $V(\Delta, \mathbb{C})$ is what remains of $\mathbb{C}P^{d+1}$ after removing all points with no zero homogeneous coordinate; in other words, it is the union of the $d+2$ standard choices for "hyperplanes at infinity".

4. Final remarks

(4.1) There is a large literature on f-vectors of simplicial complexes (defined in Section 2), see e.g. Stanley [9]. In this area the h-**vector**

$$h(\Delta) = (h_0, h_1, \ldots, h_{d+1}),$$

defined by

$$h_j := \sum_{i=0}^{j} (-1)^{j-i} \binom{d+1-i}{d+1-j} f_{i-1}, \tag{7}$$

plays an important role. Here we assume $\dim \Delta = d$ and put $f_{-1} := 1$. Knowing the h-vector of Δ is equivalent to knowing the f-vector. Equations (1) and (2) show that knowing the f^*-vector of Δ also gives equivalent information. In fact, comparison of equations (1) and (7) shows that $f^*(\Delta)$ and $h(\Delta)$ are related to $f(\Delta)$ in formally very similar fashion, namely by an upper-triangular resp. lower-triangular matrix of similar structure. Solving for $h(\Delta)$ directly in terms of $f^*(\Delta)$ one gets

$$h_j = (-1)^j \binom{d+1}{j} + (-1)^{j+1} \sum_{i=0}^{d+1-j} \binom{d-i}{j-1} f_i^*, \tag{8}$$

for $j = 1, \ldots, d+1$. In the Cohen–Macaulay case information equivalent to either of $f(\Delta)$, $h(\Delta)$ or $f^*(\Delta)$ is given also by the Betti numbers of the complex projective variety $V(\Delta, \mathbb{C})$, as shown by Proposition 3.4.

(4.2) Suppose that the Euler characteristic of $\mathrm{lk}_\Delta(\sigma)$ equals the Euler characteristic of a sphere of the same dimension for all faces $\sigma \in \Delta$ of dimension $\geq k$. This is true, for instance, for all triangulations of manifolds (with $k = 0$). Combinatorially the condition is equivalent to demanding that $\mu(\sigma, \hat{1}) = (-1)^{\operatorname{corank} \sigma}$ in the face lattice of Δ for all faces σ of dimension $\geq k$. In this case we get an alternative expression for part of the f^*-vector:

$$f_j^* = \sum_{i=j}^{d} (-1)^{d-i} f_i, \quad j = k, \ldots, d. \tag{9}$$

Namely,

$$\begin{aligned} f_j^* &= \chi\left(\Delta^{\geq j}\right) = 1 + \mu\left(\hat{0}, \hat{1}\right) = 1 - \sum_{\hat{0} < x} \mu\left(x, \hat{1}\right) \\ &= - \sum_{\hat{0} < x < \hat{1}} (-1)^{\operatorname{corank} x} = f_d - f_{d-1} + \ldots + (-1)^{d-j} f_j. \end{aligned}$$

Equating the two expressions (1) and (9) for f_j^* we get relations

$$\sum_{i=j}^{d} \left[(-1)^{i-j} \binom{i}{j} + (-1)^{d-i+1} \right] f_i = 0, \tag{10}$$

satisfied by the f-vectors of this class of complexes for all $k \leq j \leq d$. For $k = 0$ and d odd these relations are equivalent to the Dehn–Sommerville equations; see e.g. [10, p. 136]. For $k = 0$ and d even the relations (10) have to be augmented by $\chi(\Delta) = 2$ to obtain the Dehn–Sommerville equations.

(4.3) The usual functional equation (for smooth varieties) between the values at t and at $(q^d t)^{-1}$, see e.g. [4], rarely holds for the zeta function $Z_\Delta(q,t)$ of the singular varieties $V(\Delta, F)$. However, there sometimes is a "functional equation" of sorts between the values of its derivative at q and $1-q$. Namely, for any simplicial d-manifold Δ with zero Euler characteristic one has

$$(1-q)\, Z'_\Delta(q,0) = (-1)^{d+1}\, q\, Z'_\Delta(1-q,0), \tag{11}$$

where $Z'_\Delta(q,t)$ denotes the derivative of $Z_\Delta(q,t)$ with respect to t. Indeed, using Theorem 2.2 we get $Z_\Delta(q,t) = 1 + f^*(q)t + O\left(t^2\right)$, where $f^*(z) = f_0^* + f_1^* z + \cdots + f_d^* z^d$, and by (1): $zf^*(1-z) = f(z)$, where $f(z) = f_0 z - f_1 z^2 + f_2 z^3 + \cdots + (-1)^d f_d z^{d+1}$. So (11) is equivalent to $f(1-q) = (-1)^{d+1} f(q)$, i.e. to the Dehn–Sommerville equations (10). See also [6] for related remarks.

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