Tweaking

January 31, 2013. The one-dimensionality of language is constraining. To exploit that sheet of paper better we are all given to sketching, and none more so than topologists. So I was amazed when I first came to know, about 42 years ago, that some of the finest discoveries in topology had been made by someone who had lost his eyesight at 14. Indeed, Pontryagin went on to leave his mark on all areas of 20^{th} century mathematics, and also once wryly recalled that, the more he had worked on his book, the more it had seemed to shrink!

Pontryagin's homology. As in the previous note, i-chains shall be i-dimensional compact and oriented smooth sub *manifolds* M^i, for example, an i-disk is one, and—very conveniently!—the empty set 0 is one for all i. As before, we shall not define an addition on i-chains, but use the oriented *manifold-boundary* ∂M^i consisting of all points of M^i not lying in the interior of any i-disk contained in it. If $\partial M^i = 0$, then M^i is called a *closed manifold*; otherwise it is known that ∂M^i is a nonempty (i-1)-dimensional closed and oriented manifold; equivalently $\partial\partial = 0$ always. However we saw that the quotient set $H_i^{smooth}(\ ; Z)$, of the set of all closed i-manifolds, obtained by identifying all manifold-boundaries with 0, can be huge; so some 'tweaking' was needed ... *We'll put* $M^i \sim N^i$ *iff* M^i *and* N^i *are disjoint closed* i-*dimensional oriented sub manifolds whose union is the oriented manifold-boundary of an* (i+1)-*dimensional compact and oriented sub manifold.* Clearly these **moves** $M^i \sim N^i$ are symmetric, but neither reflexive nor transitive. So we proceed to the generated equivalence relation, that is, *two closed* i-*dimensional oriented sub manifolds shall be deemed equivalent iff they are related by a finite sequence of moves.* Dividing out the set of all closed i-manifolds by this relation gives $H_i^{Pontryagin}(\ ; Z)$, which once again is only a *set* with a distinguished element, but obviously can be much smaller.

If a closed sub manifold M^i of U is nonzero in ordinary homology $H_i(U;Z)$ then so is it in $H_i^{Pontryagin}(U;Z)$. However R^∞ is contractible, yet *the complex projective plane* CP^2 *is nonzero in* $H_4^{Pontryagin}(R^\infty; Z)$. Indeed Pontryagin discovered that, there is associated to any oriented smooth closed 4-manifold in a natural way an integer $p_1[M^4]$, which reverses its sign under each move, but one has $p_1[CP^2] = 1$. It is on non-bounding manifolds related by an *even number of moves* that his invariant is single-valued. The space R^∞ is very *roomy*, any two such *demi-classes* contain disjoint closed manifolds whose union and cartesian product are also in it. These binary operations on demi-classes give us the *intrinsic homology ring* Ω *of* R^∞. Likewise, considering for example the cartesian product of a polyhedron X and R^∞ gives *a natural spectral sequence with second term* H(X;Ω) which converges to the *intrinsic homology* -- this homology satisfies all the axioms of Eilenberg and Steenrod except the dimension axiom -- *group* $IH^{Pontryagin}(X; Z)$ *of* X, etc.

However much less is known about these moves if the ambient space is *not* so roomy, for example, it seems that the *full* computation of $H_i^{Pontryagin}(R^n ; Z)$ for all $n < \infty$ is beyond the present state of our art. So in the rest of this paper we'll be mostly playing around with the above moves in 2-dimensional spaces only ... and, as an added motivation for doing this, we recall from the last paper how central projection had associated to strings in 3-space some configurations of circles on the 2-dimensional screen of the observer's spherical microscope.

We'll start proceedings with some remarks on the *classification of cartesian strings.* If an orientation preserving diffeomorphism of *the plane* R^2 throws some strings M^1 onto N^1, then it preserves the sense ± of each circle, takes *adjacent*—i.e. joinable by an arc not meeting any other circle—*circles* of M^1 to adjacent circles of N^1, and maps circles bounding the infinite component of the complement of M^1 onto similar circles of N^1. In other words, we have an isomorphism between the *adjacency graphs* $G(M^1)$ and $G(N^1)$ of these planar strings, which preserves the sense and the *depth*—i.e. the edge-distance to a vertex (circle) on the infinite component of the complement—of each vertex. Further, if we delete from $G(M^1)$ all edges joining vertices of the same depth, we obtain a union of *trees* $T(M^1)$—an example is shown in Figure 1—and this graph isomorphism is determined by its restriction to these trees. Conversely, any such graph isomorphism can be realized by a sequence of irrotational planar motions.

Likewise, an orientation preserving diffeomorphism of *the* 2-*sphere* S^2 (or for that matter of any surface) which throws some strings M^1 onto N^1 gives an isomorphism of their adjacency graphs $G(M^1)$ and $G(N^1)$. Now the complement of M^1 does not have that distinguished infinite component, but M^1 certainly has an *empty circle*—i.e. a circle such that the remaining circles are all on the same side—say c_M , and this must image to an empty circle c_N of N^1. We note that if rotational motions are allowed then circles may even get reversed, for example, any great circle through the poles gets reversed every 12 hours. So we'll take care to first project M^1 and N^1 on the plane from corresponding points x_M and x_N in the complements, say, in the empty disks bounded by c_M and c_N , before we allot signs ± (anticlockwise/clockwise) to their circles, so these signs depend on the choice of c_M. The graph isomorphism preserves these signs, and transforms *the edge-distance to* c_M into the edge-distance to c_N. Using this distance we obtain as before a *pointed tree* $T(M^1, c_M)$—an example is shown in Figure 1 for the spherical strings obtained by adding a point x at infinity—and the graph isomorphism is determined by its restriction to it. Conversely, it can be shown that any graph isomorphism of the type just described can be realized by a sequence of irrotational spherical motions.

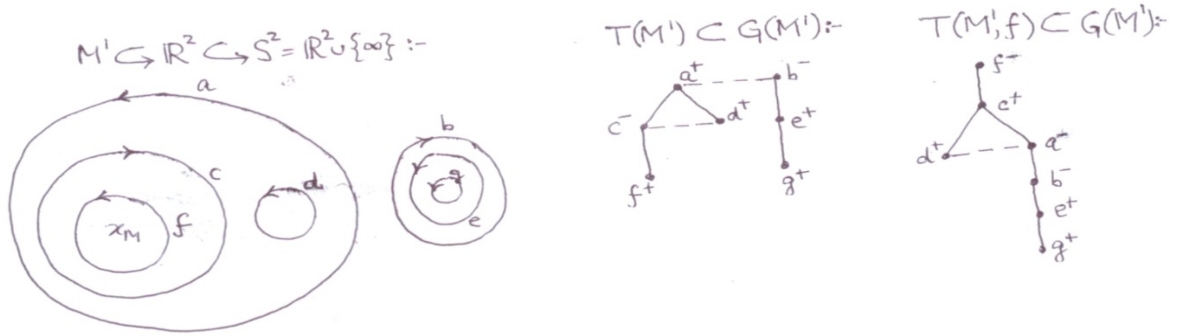

Figure 1

This absence, of a canonical way of threading the adjacency graph of spherical strings by some trees, is tied closely to the fact that, *formal reasoning is dependent on the position of the reader if it is written on a* 2-*sphere instead of the customary plane*! Customarily, in fact, formal texts are printed one-dimensionally on the line, but as G Spencer Brown's work shows, we can make the unappealing task of "unpeeling those onions" much more appealing by the very simple device of *completing each pair* ( ) *of parentheses in the line to a circle in the plane*! Thus there is a natural connection between formal texts and *unoriented strings*, a connection that not only makes the logical calculus of Leibniz and Boole much easier, but suggests many more generalizations of the same, than if we had remained within the strait-jacket of a single dimension. For example we have now, besides its one-point compactification—which after all was available even in dimension one—a host of 2-manifolds other than S^2 to play around with, and we can muse about how we should go about reading formal texts that come written on these surfaces of higher genus ...

So logic gives us an added incentive for making *mod* 2 *calculations*—i.e. those in which one ignores differences of orientation—which anyway we all make first because they are so much easier! For example, any planar M^1 is the mod 2 boundary of the disjoint regions between its circles of depth 0 and 1, between its circles of depth 2 and 3, ... , and the disks bounding the remaining empty circles. Thus $H_1^{\text{Pontryagin}}(R^2;Z/2) = 0$ *by a single* mod 2 *move*. On the other hand when we are keeping track of orientation two concentric clockwise circles do not form a manifold-boundary. But consider the move in which we thicken the outer circle to a thin oriented annulus and simultaneously cap the inner circle by an oriented 2-disk. We are left with a single anticlockwise circle which we can now cap with an oriented 2-disk. Generalizing this argument in the obvious way we see that $H_1^{\text{Pontryagin}}(R^2;Z)$ *is also zero but there are planar strings* M^1 *which may require an arbitrarily large number of moves before they can be killed off*. For example, if all circles of M^1 are clockwise, and it has an empty circle of depth t, then we'll need at least t+1 moves.

Intrinsic homology is also called *oriented cobordism*. If the ambient space is oriented, saying that a sub manifold is oriented is the same as saying that its normal bundle is oriented. Which, for codimension one sub manifolds, is the same as saying that the sub manifold is *framed*—i.e. that a trivialization of its normal bundle is given—but in bigger codimensions being framed is a stronger requirement. One defines *framed cobordism* exactly as before using now framed sub manifolds and the framing induced on the manifold-boundary. A construction of Pontryagin's gives *a natural bijection between framed cobordism classes of codimension* p *closed sub manifolds and homotopy classes of maps into the* p-*sphere*. So, in particular, if the ambient space is an n-ball—the sub manifolds are now taken in its interior—we are talking of the nth *homotopy group* of the p-sphere. Though much is known about these groups, to the best of my knowledge *all* the homotopy groups of the 2-sphere are still not known. Anyway, we see that $IH_1^{Pontryagin}(V^2;Z)$ *of a surface* V^2 identifies with homotopy classes $[V^2,S^1]$, which under complex multiplication of the circle is isomorphic to the free abelian integral cohomology $H^1(V^2;Z)$; likewise, $H_1^{Pontryagin}(V^2;Z/2)$ is isomorphic to $H_1(V^2;Z/2)$.

Spencer Brown's homology. We define $H_i^{SB}(\ ;Z)$ and $H_i^{SB}(\ ;Z/2)$—for all n-dimensional manifolds—exactly like Pontryagin's homology, except that *we disallow the use of* n-*disks* in the moves.

We assert that $H_1^{SB}(R^2; Z/2) = \{0,1\}$. More precisely that any *expression*, i.e. a planar configuration of unoriented circles, can be transformed by finitely many allowed moves—i.e. combinations of the fission/fusion and perturbation shown in Figure 2—into the expression having no circle (these form the equivalence class 0 and may be called *false*) or can be transformed into an expression having one circle (these form the equivalence class 1 and may be called *true*) but not both. Indeed, if all circles are of depth zero, they can be replaced by a single new circle encircling them. Otherwise, we pick a maximal bunch of maximal depth circles which are within the same circle of depth one less. We can now erase the mentioned circles all at once—and simultaneously perturb the remaining circles slightly—for they together form the mod 2 boundary of a compact 2-manifold which is not a 2-disk. Continuing this process we would either have erased all circles or would be left with a configuration of depth zero only. That one circle cannot be transformed into no circle follows from the fact that the planar complement of a point is homologically non-trivial.

Figure 2

However this feat can be achieved on the 2-sphere: we fission the circle into two, and then bound these two by a compact sub manifold of the 2-sphere which is not a 2-disk. So $H_1^{SB}(S^2; Z/2)$ *is trivial*—and more generally $H_1^{SB}(V^2;Z/2)$ is isomorphic to $H_1(V^2;Z/2)$ for any closed surface V^2—*which again shows that for formal text written on the* 2-*sphere the position* x *of the reader is important*. Once that is known we use relative homology $H_1^{SB}(S^2,x; Z/2)$—this is defined using only expressions that are written in the complement of x—which is isomorphic to $H_1^{SB}(R^2; Z/2)$, so we can exactly as before dub each expression in the complement of x as x-*true* or x-*false* but not both.

More generally, we can not only consider configurations of circles on any surface V^2, but also sprinkle finitely many *variables* in their complement. Thus obtaining *statements*, written on V^2 about these variables, which are deemed to be unknown (planar) expressions. Using surfaces of positive genus and homologically non-trivial circles we can thus even rehabilitate the self-referential statements that are taboo in planar logic! Further, it is meaningful to ask for any statement on V^2 whether it belongs to the same homology class of say $H_1^{SB}(V^2,x;Z/2)$ for all values of the variables—in two-valued logic these are called tautologies/oxymorons—or whether this *truth-value* can vary, etc.

Notes

1. The definitions in Milnor, *A survey of cobordism theory,* L'Enseignement Math 8 (1962) 16-23 are for abstract manifolds, i.e. sub manifolds of roomy spaces. If n > 2(i+1) then $IH_i^{Pontryagin}(R^n ; Z) = IH_i^{Pontryagin}(R^\infty; Z) = \Omega_i$ and Theorem 2' of this survey tells us that *modulo* 2-*torsion our* Ω *is an integral polynomial ring with one generator in each dimension divisible by four*; the 4-dimensional generator arising of course from Pontryagin's watershed discovery that CP^2 is not a manifold-boundary, which appeared in print first in 1947 in his great paper on *Characteristic cycles of differentiable manifold*; and this original proof—which suggests strongly that what is going on here is a quaternionic analogue of Cauchy's calculus of residues for complex numbers—is still worth reading.

2. In this 1947 paper was posed also that bewitching problem of finding an algorithmic definition of *pontryagin numbers* for all combinatorial manifolds ... and it seems that at long last we are now almost there ... Anyway, *Gaifullin's homology* is also algorithmic much like Khovanov's ... and $p_1[CP^2] = 1$ is also a statement about strings ... because, if we partition the boundary of all quaternions of size ≤ 1 into circles under the action of the complex numbers of size 1, then CP^2 is exactly what we get from this 4-dimensional ball if we squash each of these circles to a point.

3. A general version of the cited spectral sequence is given in Graeme Segal's *Classifying spaces and spectral sequences,* Publications I.H.E.S. 34 (1968) 105-112, a paper making "no great claim to originality" which brought to light Grothendieck's graph theoretic—or if you prefer category theoretic—homologies, which let us exploit the defining moves of equivalence relations to get more juice from the information that is available.

4. Any graph is the adjacency graph of strings on a surface of a large genus, but graphs of planar/spherical strings are special, e.g., deleting any non-empty circle disconnects them; so it is likely they can be characterized; and maybe even counted off ... for ever since Cayley counting all sorts of trees has been a fairly popular pastime ...

5. We note that Pontryagin's homology of planar strings is zero even if we insist that only those 2-dimensional sub manifolds be used which are oriented compatibly with a given orientation of R^2 : for we can always thicken an empty circle, which reverses its orientation, and then kill it off. That just the slightest thickening reverses the orientation again points towards that *uncertainty* in our knowledge about the observed orientation of strings.

6. The inverse image of a regular value of a smooth map $V^{n+p} \times [0,1] \rightarrow S^p$ gives us a framed (n+1)-dimensional sub manifold of the cylinder ending in framed n-dimensional sub manifolds M^n and N^n of V^{n+p}. So in the concluding section, "§ 7. *Framed cobordism and the Pontryagin construction*," of that thin book I cited before, Milnor finds it convenient to adopt the statement just made as his definition of "M^n *and* N^n *are framed codordant in* V^{n+p}," but it can be shown that this definition is equivalent to the definition by moves that we have preferred in the text.

7. My $H_1^{SB}(R^2; Z/2) = \{0,1\}$ is merely a reformulation of Chapter 4 of that wonderful book by G Spencer Brown, *Laws of Form* (1972), which I received as a gift from L H Kauffman about 12 years ago after I pointed out a slip in his efforts to revive an unpublished proof of the four colour theorem by Spencer Brown. Had the latter used some logic on a closed surface of a possibly higher genus to prove this theorem about the surface of genus zero? Is it my imagination only, or is Mochizuki also trying to do these days something similar, as he goes about hunting for ABC ...

8. I've still not found the time to reread, let alone finish, that paper on *Bina's garden* ... likewise, the much longer and more ambitious one before that on *Hyperbolic manifolds* ... but anyway I've managed to not only reread but polish and type up these notebook scribblings from the last four weeks ... so maybe there is hope for me yet ...