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TVERBERG PARTITIONS AND BORSUK-ULAM THEOREMS

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An N-dimensional real representation E of a finite group G is said to have the "Borsuk-Ulam Property" if any continuous G-map from the $(N+1)$-fold join of G (an N-complex equipped with the diagonal G-action) to E has a zero. This happens iff the "Van Kampen characteristic class" of E is nonzero, so using standard computations one can explicitly characterize representations having the B-U property. As an application we obtain the "continuous" Tverberg theorem for all prime powers q, i.e., that some q disjoint faces of a $(q-1)(d+1)$-dimensional simplex must intersect under any continuous map from it into affine d-space. The "classical" Tverberg, which makes the same assertion for all linear maps, but for all q, is explained in our set-up by the fact that any representation E has the analogously defined "linear B-U property" iff it does not contain the trivial representation.

1. Introduction.

This paper is essentially an analysis of a method which I had used in a manuscript [**19**] circulated in 1988-89. Some of its results have in the meantime been independently obtained by others, and it is possible that the newer methods of [**21**] might lead to better results. Nevertheless, it does give a complete account of one aspect of *"the method of deleted joins"*: it delineates clearly its power and limitations, as far as the two topics mentioned in the title are concerned, *if one uses only finite groups,* as against [**21**], where we use a continuous group action.

In 1966 Tverberg [**24**] showed that any cardinality $(q-1)(d+1)+1$ subset of a real affine d-dimensional space can be partitioned into q disjoint subsets whose convex hulls have a nonempty intersection; a much easier proof is given in [**20**]. There is a "continuous" analogue which asks more: given any continuous map f from a $(q-1)(d+1)$-simplex into d-space, can one always find q disjoint faces $\sigma_1, \dots, \sigma_q$ of this simplex such that $f(\sigma_1) \cap \ldots \cap f(\sigma_q)$ is nonempty? For q *prime* this was established by Bárány-Shlosman-Szücs [**4**]. In [**18**] I gave an easy proof of this result using a deleted $\mathbb{Z}/q$-join of the N-simplex, $N = (q-1)(d+1)$, viz. the $(N+1)$-fold join $E_N(\mathbb{Z}/q) = \mathbb{Z}/q \cdot \ldots \cdot \mathbb{Z}/q$.

In [19] I attempted to generalize this argument to all q by using, in addition, the "Van Kampen obstruction" class e.

The importance of this characteristic class $e(\mathbb{E}) \in H^N(G, \widehat{\mathbb{Z}})$, $n = \dim(\mathbb{E})$, which we define in 2.6 for any real representation $\mathbb{E}$ of any finite group G, stems from the fact — see Theorem 1, 2.6.2 — that it is nonzero iff $\mathbb{E}$ has the **Borsuk-Ulam property**, i.e., any continuous G-map $E_N(G) \to \mathbb{E}$ has a zero. Using the argument of [18], the "continuous" Tverberg holds if one has an order q group G for which $\mathbb{L}^{\perp}(G)$, the $(d+1)$-fold direct sum of the non-trivial part of the regular representation, has this B-U property. Our Theorem 2, 2.6.3 gives a complete characterization of complex $\mathbb{Z}/q$-representations having the B-U property. In particular, it shows that the representations $\mathbb{L}^{\perp}(\mathbb{Z}/q)$ all have this property iff q is prime, which gives of course the B-S-S theorem, and shows also that to go beyond one needs to look at finite non-cyclic groups. Amusingly, the original Tverberg theorem also fits neatly into this B-U framework: we check that the argument of [20] or [10] is really just the same, except that one now invokes *a linear analogue* 2.4 *of the B-U property* which holds for all q. The next Theorem 3, 2.8.1 generalizes the "continuous" Tverberg to all *prime powers* $q = p^k$ and has also been proved independently by Ozaydin [16] and Volovikov [25]. It follows at once from Theorem 4, 2.8.2 which says that a representation of $(\mathbb{Z}/p)^k$ has the B-U property iff it does not contain the trivial representation. Finally in 2.9, we embed the $\mathbb{Z}/q$-action of $\mathbb{L}^{\perp}(\mathbb{Z}/q)$ in an action of the symmetric group Σ_q, and show — see Theorem 5, 2.9.3 — that the characteristic class of this Σ_q representation is zero iff q is not a prime power. To go beyond prime powers it thus seems necessary to use continuous group actions.

The exposition below is self-contained except that we refer to the literature for standard facts regarding Chern classes of finite group actions. For more background material see also Mark de Longueville's notes [13] of a seminar based on this paper.

2. Borsuk-Ulam representations.

The main character of our story is a real N-dimensional group representation $\mathbb{E}$ which does not contain the trivial representation, mostly $\mathbb{E} = \mathbb{L}^{\perp}$ (defined in 2.2 below) which has dimension $N = (q-1)(d+1)$.

2.1. By the q-th *deleted join* [17] $K * \cdots * K$ of a simplicial complex K one understands the subcomplex of its q-fold join $K \cdot \ldots \cdot K$ consisting of all simplices $(\sigma_1, \ldots, \sigma_q)$ with $\sigma_i \cap \sigma_j = \emptyset \;\forall\, i \neq j$. Mostly $K = [N] =$ all faces of the N-simplex $\{e_1, \ldots, e_{N+1}\}$. Let Q be a cardinality q set. Denoting the q copies of each e_α by ge_α, $g \in Q$, $[N] \cdot \ldots \cdot [N]$ consists of all subsets of the cardinality $q(N+1)$ set $\{ge_\alpha : g \in Q,\ 1 \leq \alpha \leq N+1\}$, and $[N] * \cdots * [N]$ of all faces of all N-simplices of the type $\{g_1e_1, \ldots, g_{N+1}e_{N+1}\}$. So $[N] * \cdots * [N]$ (q times) identifies with $E_N(Q) = Q \cdot \ldots \cdot Q$ ($N+1$ times).

Frequently we'll equip the set Q with a group structure G, and then let G act simplicially on $[N] \cdot \ldots \cdot [N]$ by $h \bullet (ge_\alpha) = (hg)e_\alpha$. Note that this action preserves, and is free on, the subcomplex $[N] * \cdots * [N]$. We recall that such free G-complexes $E_N(G) = G \cdot \ldots \cdot G$ ($N+1$ times), $EG = \cup_N E_N(G)$, go into Milnor's definition [**14**] of a *classifying space* BG *of* $G : BG = EG/G = \cup_N(B_N G)$, where $B_N G = E_N(G)/G$.

2.2. We'll identify our affine d-space $\mathbb{A}^d$ with the hyperplane $\Sigma_k x_k = 1$ of $\mathbb{R}^{d+1}$, and the q-fold product $\mathbb{R}^{d+1} \times \cdots \times \mathbb{R}^{d+1}$ with the vector space $\mathbb{L}$ of all real $(d+1) \times q$ matrices, with $\mathbb{L}^\perp$ denoting the $(q-1)(d+1)$ dimensional subspace consisting of all matrices having row sums zero. Note that $\mathbb{L}^\perp$ is the orthogonal complement of the diagonal subspace Δ of matrices having all columns equal to each other.

We'll index the columns of our matrices by the cardinality q set Q. Frequently Q will be equipped with a group structure G, and then we'll permute the columns by left translations. The resulting representations of G will be denoted $\mathbb{L}(G)$ and $\mathbb{L}^\perp(G)$. Note that $\mathbb{L}(G) = \mathbb{R}^{d+1}[G]$, the $(d+1)$-fold direct sum of the *regular representation* $\mathbb{R}[G]$ provided by each row, and that $\mathbb{L}^\perp(G)$ contains no trivial representation. So the action of G on the unit sphere $S(\mathbb{L}^\perp)$ is always without fixed points. When $d+1$ is even we'll identify $\mathbb{L}(G)$ with the representation $\mathbb{C}^{(d+1)/2}[G]$ provided by all $\frac{d+1}{2} \times q$ complex matrices by taking real and imaginary parts of each row, and we'll equip $\mathbb{L}(G)$ with the orientation prescribed by this complex structure.

For the case $G = \mathbb{Z}/q$ note that the action is free on $S(\mathbb{L}^\perp)$ iff q is prime, and that the action preserves the orientation of $\mathbb{L}^\perp(\mathbb{Z}/q)$ iff $(q-1)(d+1)$ is even.

2.3. Proof of theorems of Tverberg and Bárány-Shlosman-Szücs. Let s_α, $1 \leq \alpha \leq N+1$, $N = (q-1)(d+1)$, be the points of the given set $S \subset \mathbb{A}^d$ and consider the linear map $K = [N] \xrightarrow{f} \mathbb{A}^d$ such that $e_\alpha \mapsto s_\alpha \,\forall\, \alpha$. More generally consider any continuous map $[N] \xrightarrow{f} \mathbb{A}^d$. We want to show that there exist q disjoint faces $\sigma_1 \ldots, \sigma_q$ of K such that $f(\sigma_1) \cap \ldots \cap f(\sigma_q) \neq \emptyset$. Equivalently, if we compose the q-fold join $K * \cdots * K \to \mathbb{A}^d \cdot \ldots \cdot \mathbb{A}^d \subset \mathbb{R}^{d+1} \times \cdots \times \mathbb{R}^{d+1} = \mathbb{L}$ of f with the orthogonal projection $\mathbb{L} \to \mathbb{L}^\perp$ to get a map

$$s : [N] * \cdots * [N] \to \mathbb{L}^\perp,$$

then what we have to show is that $0 \in \mathrm{Im}\,(s)$.

For this, note first that s commutes with the group actions, defined above. Now the linear case follows by applying the "linear Borsuk-Ulam" theorem 2.4. Likewise, for q prime, we see that the $\mathbb{Z}/q$-map s associated to a continuous f must have a zero, by using the generalization 2.5 of the usual continuous Borsuk-Ulam. □

2.4. "Linear Borsuk-Ulam". *If $\mathbb{E}$ does not contain the trivial representation, then any linear G-map $s : E_N(G) \to \mathbb{E}$ has a zero.*

Note that the condition on $\mathbb{E}$ is obviously necessary.

Proof. This is a particular case of Bárány [3] the argument being as follows. If conv $\langle s(g_\alpha e_\alpha) : g_\alpha \in G, 1 \le \alpha \le N+1\rangle$ is at a distance $\delta > 0$ from $0 \in \mathbb{E}$, then its nearest point P is contained in the hyperplane H normal to $0P$ and out of the points $s(g_\alpha e_\alpha)$ we can choose $\le N$ which all lie on H and are such that P is in their convex hull. The remaining points will be either on H or in the component of $\mathbb{E}\backslash H$ not containing $\{0\}$. Let $s(g_\beta e_\beta)$ be any of these points. Since s commutes with the G actions, and $\mathbb{E}$ does not contain the trivial representation, we have $\Sigma_g s(g\, e_\beta) = \Sigma_g g(s\, e_\beta) = 0$. So some $s(g\, e_\beta)$ must be in the component of $\mathbb{E}\backslash H$ which contains $\{0\}$. Replacing g_β by such a g we can make δ still smaller. So the minimum δ must be zero. □

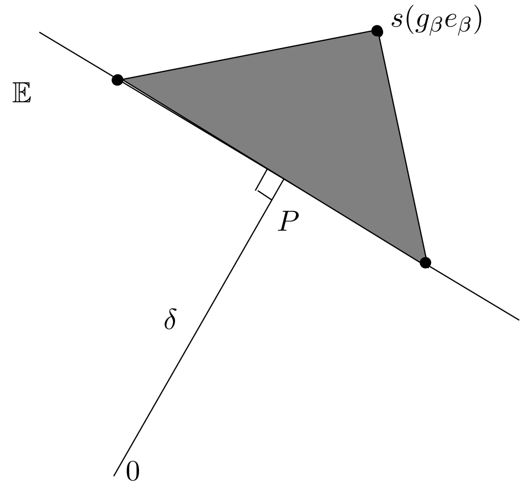

Figure 1.

2.5. Liulevicius [12], Dold [8]. *If $G \neq 1$ acts freely on $S(\mathbb{E})$ then $\mathbb{E}$ has the Borsuk-Ulam property, i.e., every continuous G-map $s : E_N(G) \to \mathbb{E}$ has a zero.*

This generalizes Borsuk's theorem [6] which says (because $E_N(\mathbb{Z}/2) =$ octahedral N-sphere equipped with the antipodal $\mathbb{Z}/2$ action) that the representation of $\mathbb{Z}/2$ in $\mathbb{R}^N$ given by $x \mapsto -x$ has the B-U property.

Proof. It suffices to prove the result for complex representations, for if there were a G-map $E_N(G) \to S(\mathbb{E})$, then its 2-fold join would provide a G-map $E_{2N}(G) \subset E_{2N+1}(G) \to S(\mathbb{E}) \cdot S(\mathbb{E}) = S(\mathbb{E} \oplus \mathbb{E} \cong \mathbb{E} \otimes \mathbb{C})$ with G acting freely on $S(\mathbb{E} \otimes \mathbb{C})$.

Also it suffices to do just the prime cyclic case: for each G contains a subgroup $H \cong \mathbb{Z}/p$, and this case then gives us at least $|G| \div p$ zeros, one in each $E_N(H_g) = (H_g) \cdot \ldots \cdot (H_g)$. So the result follows from 2.6.3 which in fact gives for all q an explicit characterization of complex $\mathbb{Z}/q$ representations having the Borsuk-Ulam property. $\square$

2.6. Characteristic classes of representations. Recall that the cohomology of $G \cong \pi_1(BG)$ is defined to be that of the classifying space BG. Likewise — see the appendix of Atiyah [**1**] — the characteristic classes of any representation $\mathbb{E}$ of G are defined to be those of the corresponding vector bundle $\mathcal{E} = EG \times_G \mathbb{E} \to BG$.

2.6.1. *In dimensions $\leq N$ a characteristic class of $\mathcal{E}$ vanishes iff its restriction to $B_N G$ vanishes.*

Proof. "Only if" is obvious. Using naturality of characteristic classes note that the restriction is the corresponding class of the bundle $E_N G \times_G \mathbb{E} \to B_N G$. Further the $(N+1)$-fold join $E_N G = G \cdot \ldots \cdot G$ is $(N-1)$-connected, so its identity map extends to a continuous G-map $(EG)_N \to E_N G$ from the N-skeleton $(EG)_N$ of EG to $E_N G$, thus giving us a bundle map $(EG)_N \times_G \mathbb{E} \to E_N G \times_G \mathbb{E}$. So, again by naturality, the corresponding class of $(EG)_N \times_G \mathbb{E} \to (BG)_N$ is also zero. This gives "if" because the inclusion induced map $H^i(BG) \to H^i((BG)_N)$ is injective for $i \leq N$. $\square$

We'll equip $\mathbb{E}$ with some orientation and let $\widehat{\mathbb{Z}}$ denote the integers equipped with the G-action $g \bullet n = \pm n$, the sign depending on whether $E \xrightarrow[\cong]{g} \mathbb{E}$ preserves or reverses orientation. Now take any continuous G-map $s : EG \to \mathbb{E}$ with no zeros on the $(N-1)$-skeleton and associate to any oriented N-simplex σ the *degree* of the map $s : \partial\sigma \to \mathbb{E}\backslash\{0\}$. This cochain $\sigma \mapsto \deg(s|\partial\sigma)$, which is equivariant with respect to the G-actions of EG and $\widehat{\mathbb{Z}}$, can be verified to be a cocycle, and its cohomology class $e(\mathbb{E}) \in H^N(G; \widehat{\mathbb{Z}})$ verified to be independent of the map s chosen. For these standard facts of *obstruction theory* see Steenrod [**23**, §35].

For example, we can choose s linear, when of course $\deg(s|\partial\sigma) \in \{-1, 0, +1\}$, and the "Linear Borsuk-Ulam" 2.4 tells us that this cocycle is nonzero for all $\mathbb{E}$ not containing the trivial representation. The vanishing of its cohomology class interprets as follows.

2.6.2.

Theorem 1. *The representation $\mathbb{E}$ has the Borsuk-Ulam property iff the characteristic class $e(\mathbb{E}) \in H^N(G; \widehat{\mathbb{Z}})$ is nonzero.*

Proof. By 2.6.1 this class is zero iff the corresponding class of the bundle $E_N G \times_G \mathbb{E} \to B_N G$ is zero, but this happens (see [**23**, §35]) iff this vector bundle admits a continuous nonzero section, i.e., iff there is a continuous G-map $E_N G \to \mathbb{E}$ having no zeros. □

It might be appropriate to call $e(\mathbb{E})$ the *van Kampen class* of $\mathbb{E}$ because it can be traced back, for the case $G = \mathbb{Z}/2$ to [**11**]. In case the action of G on $\mathbb{E}$ is orientation preserving, i.e., $\widehat{\mathbb{Z}} = \mathbb{Z}$, the integers equipped with the trivial action of G, then $e(\mathbb{E}) \in H^N(G;\mathbb{Z})$ identifies — see Milnor-Stasheff [**15**, p. 147] — with the *Euler class* of the oriented N-dimensional plane bundle $\mathcal{E} \to BG$. Thus, if N is even and $\mathbb{E}$ is a complex $N/2$-dimensional representation of G, then $e(\mathbb{E})$ coincides — see [**15**, p. 158] — with the $N/2$-th *Chern class* $c_{N/2}(\mathbb{E})$ of this complex $N/2$-dimensional bundle $\mathcal{E}$. Evens [**9**] has shown that $c_{N/2}(\mathbb{E})$ *can always be computed purely algebraically*, provided one knows the cohomology ring of G and the Brauer decomposition of $\mathbb{E}$. These computations can be quite hard, but the simple cases we need are easily dealt with directly.

We recall that $\mathbb{Z}/q$ has q irreducible complex representations, all one-dimensional, being in fact the q homomorphisms $\mathbb{Z}/q \to \mathbb{C}^\times$, $\omega \mapsto \omega^\ell$, $1 \leq \ell \leq q$, where ω denotes the generator $\exp(2\pi i/q)$ of $\mathbb{Z}/q$.

2.6.3.

Theorem 2. *Let m_ℓ denote the multiplicity of ω^ℓ in the irreducible decomposition of the complex $N/2$-dimensional representation $\mathbb{E}$ of $\mathbb{Z}/q$. Then $\mathbb{E}$ has the Borsuk-Ulam property iff $q \not| \Pi_\ell(\ell)^{m_\ell}$.*

Proof. We'll use 2.6.2. The multiplicativity of Chern classes shows

$$e(\mathbb{E}) = c_{N/2}(\mathbb{E}) = \Pi_\ell(c_1(\omega^\ell))^{m_\ell},$$

where $c_1(\omega^\ell) \in H^2(\mathbb{Z}/q;\mathbb{Z})$ denotes the first Chern class of the representation $\omega \mapsto \omega^\ell$ and multiplication is the cup product of $H^*(\mathbb{Z}/q;\mathbb{Z})$. Since $c_1 : \mathrm{Hom}\,(G, \mathbb{C}^\times) \to H^2(G;\mathbb{Z})$ is always a group isomorphism — see Atiyah [**1**, (3), p. 62] — it follows that

$$e(\mathbb{E}) = \Pi_\ell(\ell c_1(\omega))^{m_\ell} = \Pi_\ell(\ell)^{m_\ell}(c_1(\omega))^{N/2} = u^{N/2}(\Pi_\ell(\ell)^{m_\ell}),$$

where $u : H^i(\mathbb{Z}/q;\mathbb{Z}) \to H^{i+2}(\mathbb{Z}/q;\mathbb{Z})$ is the map given by taking cup product with the generator $c_1(\omega)$ of $H^2(\mathbb{Z}/q;\mathbb{Z})$ and $\Pi_\ell(\ell)^{m_\ell} \in \mathbb{Z} = H^0(\mathbb{Z}/q;\mathbb{Z})$. This periodicity map u is an epimorphism for $i = 0$ and an isomorphism for $i \geq 1$ (and the remaining odd dimensional cohomology of $\mathbb{Z}/q$ is zero): see Cartan-Eilenberg [**7**, p. 260]. So it follows that $e(\mathbb{E})$ vanishes iff $u(\Pi_\ell(\ell)^{m_\ell}) = \Pi_\ell(\ell)^{m_\ell} \cdot c_1(\omega)$ vanishes, i.e., iff q divides $\Pi_\ell(\ell)^{m_\ell}$. □

It seems one can give a similar explicit characterization of the complex Borsuk-Ulam representations of any finite Abelian group G.

2.7. The last theorem gives rise to some remarks.

2.7.1. Obviously, for a complex $\mathbb{Z}/q$ representation $\mathbb{E}$, the *group action is free on* $S(\mathbb{E})$ *iff only those representations* $\omega \mapsto \omega^\ell$ *occur in it for which* ℓ *is relatively prime to* q. So 2.6.3 shows that the Borsuk-Ulam holds in many cases not covered by 2.5.

However 2.6.3 also shows that the *if* q *is composite and* $d+1$ *is an even number* ≥ 4, *then there exist continuous* $\mathbb{Z}/q$ *maps* $E_N(\mathbb{Z}/q) \to \mathbb{L}^\perp$ *having no zeros.* This follows because, for $\mathbb{L}^\perp = \mathbb{C}^{(d+1)/2}[\mathbb{Z}/q]$ the number $\Pi_\ell(\ell)^{m_\ell}$ equals $((q-1)!)^{(d+1)/2}$, and $q|(q-1)!$ unless q is prime or equal to 4. Thus to generalize the continuous version of the proof of 2.3 beyond the case q prime one needs non-cyclic groups G.

2.7.2. Sierksma [**22**] has conjectured that a cardinality $(q-1)(d+1)+1$ subset of d-space has at least $((q-1)!)^d$ Tverberg partitions, i.e., that the linear map $s : E_N(Q) \to \mathbb{L}^\perp$ of 2.3 has at least $((q-1)!)^{d+1}$ zeros. It may in fact be possible to algebraically count these generic *Tverberg zeros* with appropriate *local degrees* ± 1, so that one always get $((q-1)!)^{d+1}$. One cannot hope however for a similar index formula for *Tverberg partitions*, because this would imply, for $q = 3$ and $d = 2$, that the number of these partitions is always even, which is not so.

If one attempts such a signed counting by using finite group actions then one runs into problems. For example by taking $S \subset \mathbb{A}^d$ in a *general position* we can ensure that s has no zeros on the $(N-1)$-skeleton of $E_N(\mathbb{Z}/q)$ — i.e., that no proper subset of S has a Tverberg partition into q parts — and then evaluate the cocycle $\sigma \mapsto \deg(s|\partial\sigma)$ of $e(\mathbb{L}^\perp)$ on some equivariant N-cycle of $E_N(\mathbb{Z}/q)$. However this algebraic counting does not give an *integer* invariant because $e(\mathbb{L}^\perp)$ lives in $H^N(\mathbb{Z}/q;\mathbb{Z}) \cong \mathbb{Z}/q$ and so is of finite order. Anyhow for the q prime case this method does suffice to give rough lower bounds for the number of Tverberg partitions: see Vučic-Zivaljevic [**26**].

2.7.3. Sierksma's problem is *stable with respect to* d i.e., we can increase d by 1. To see this add, to a general position $S \subset \mathbb{A}^d \subset \mathbb{A}^{d+1}$, $q-1$ new points of $\mathbb{A}^{d+1}\backslash\mathbb{A}^d$, and at the same time perturb one of the old points v out of $\mathbb{A}^d$. In a Tverberg partition of this set $\widehat{S} \subset \mathbb{A}^{d+1}$ the part containing v cannot contain any of the new $q-1$ points, for then some other part contains none and so is in $\mathbb{A}^d$, and thus restricting to $\mathbb{A}^d$ we would have got a Tverberg partition of the proper subset $S\backslash\{v\}$ of the general position set $S \subset \mathbb{A}^d$. Thus $\widehat{S}$ has at most $(q-1)!$ times as many Tverberg partitions as S. A similar but simpler argument shows likewise that the "continuous" Tverberg problem is also stable with respect to d. So we can assume $d+1$ *even* (this we'll do from here on), $d \gg q$, etc., with impunity in our proofs.

2.8. The next result has also been proved independently by Ozaydin [**16**] and Volovikov [**25**].

2.8.1.

Theorem 3. *The "continuous" Tverberg theorem is true for all prime powers $q = p^k$.*

Proof. We know from 2.7.1 that the argument of 2.3 can work for $k \geq 2$ only if we use the representation $\mathbb{L}^{\perp}(G)$ of some non-cyclic order q group G. By 2.8.2 below it does work for $G = (\mathbb{Z}/p)^k$. □

2.8.2.

Theorem 4. *The B-U property holds for any representation $\mathbb{E}$ of $(\mathbb{Z}/p)^k$ not containing the trivial representation.*

Proof. Without loss of generality (cf. proof of 2.5) we can assume $\mathbb{E}$ complex. So it is the direct sum of irreducible one-dimensional representations $(\mathbb{Z}/p)^k \to \mathbb{C}^{\times}$. These form a group, each member being of the type

$$(\omega_1, \dots, \omega_k) \mapsto \omega_1^{\ell_1} \cdots \omega_k^{\ell_k}$$

where ω_i's denote copies of the generator $\omega = \exp(2\pi i/p)$, and $0 \leq \ell < p$ with not all ℓ_i's zero. If, in the isomorphic group $H^2((\mathbb{Z}/p)^k; \mathbb{Z})$, x_i denotes the first Chern class of $(\omega_1, \dots, \omega_k) \mapsto \omega_i$, then $(\omega_1, \dots, \omega_k) \mapsto \omega_k^{\ell_1} \cdots \omega_k^{\ell_k}$ has first Chern class $\ell_1 x_1 + \cdots + \ell_k x_k$.

With mod p field coefficients the cohomology algebra of $(\mathbb{Z}/p)^k$ is isomorphic to the polynomial algebra $\mathbb{Z}/p[x_1, \dots, x_k]$ — this follows by using the case $k = 1$, $B(\mathbb{Z}/p) \times \cdots \times B(\mathbb{Z}/p) \simeq K((\mathbb{Z}/p)^k, 1)$, and the Kunneth formula for field coefficients — and so has no zero divisors. Therefore the cup product $e(\mathbb{E})$ of all these nonzero 2-dimensional classes $\ell_1 x_1 + \cdots + \ell_k x_k$ is nonzero, which by 2.6.2 is same as saying that $\mathbb{E}$ has the B-U property. □

2.8.3. *Let $\mathbb{E}$ be as above, and $\mathbb{F}$ be any other representation of $(\mathbb{Z}/p)^k$ with* $\dim(\mathbb{F}) > \dim(\mathbb{E})$. *Then there does not exist a continuous $(\mathbb{Z}/p)^k$-map from the sphere $S(\mathbb{F})$ to the sphere $S(\mathbb{E})$.*

This is another (known) generalization of Borsuk's theorem [**6**] which is the case of $\mathbb{Z}/2$ acting on two Euclidean spaces via $x \mapsto -x$. See e.g., Atiyah-Tall [**2**] and Bartsch [**5**] for more on equivariant maps between representation spheres.

Proof. Since $\dim(\mathbb{F}) > \mathbb{N}$ the connectivity of the sphere $S(\mathbb{F})$ allows us to construct a continuous $(\mathbb{Z}/p)^k$-map into it from the free N-dimensional $(\mathbb{Z}/p)^k$-complex $E_N((\mathbb{Z}/p)^k)$. This and 2.8.2 rule out the possibility of any equivariant map $S(\mathbb{F}) \to S(\mathbb{E})$. □

2.9. Unfortunately one cannot extend the "continuous" Tverberg further by a similar use of other groups G of order $q \neq p^k$.

2.9.1. *For any finite group G whose order is not a prime power there exists a continuous G-map $E_N G \to \mathbb{L}^{\perp}(G)$ having no zeros.*

One way of checking this is to note first that *if $H < G$, and $\mathbb{L}^{\perp}(H)$ does not have the Borsuk-Ulam property, then $\mathbb{L}^{\perp}(G)$ also does not have the Borsuk-Ulam property.* This follows because $\mathbb{L}(G)$ is *induced* by $\mathbb{L}(H)$, so allowing us to construct from a given H-map $E_{N'}(H) \to \mathbb{L}(H)$ whose image misses the diagonal, a G-map $E_N(G) \to \mathbb{L}(G)$ whose image also misses the diagonal. Hence by 2.6.3 we are only left to consider those G's, of non-prime power order, which are such that all elements are of prime order. Some group theory shows that such a G must contain a subgroup H which is *a non-Abelian extension of $(\mathbb{Z}/p)^k$ by a cyclic group of a different prime order.* The proof can now be completed by checking that the Euler class of $\mathbb{L}^{\perp}(H)$ is zero.

We have omitted the details — cf. Bartsch [**5**] who proceeds as above (instead of Euler classes he uses a Burnside ring argument) to obtain a similar result about maps between representation spheres — because we'll see below that a simpler reasoning gives more.

2.9.2. The point to note is that in 2.1 to 2.3 the natural group to use was the *symmetric group* Σ_q of all permutations of Q. It acts in the obvious way on $Q \cdot \ldots \cdot Q$, and on $\mathbb{L}$, and the map $s : Q \cdot \ldots \cdot Q \to \mathbb{L}^{\perp}$ of 2.3 commutes with these Σ_q actions. Further $\mathbb{L}^{\perp}$ contains no trivial representation of Σ_q, or for that matter of any subgroup of Σ_q which acts transitively on Q. The only advantage in using the simply transitive subgroups G was that their action on $Q \cdot \ldots \cdot Q$ is free.

When we consider $\mathbb{L}^{\perp}$ as a Σ_q-representation its Euler class lives in $H^N(\Sigma_q; \mathbb{Z})$. We were previously looking at its restrictions to $H^N(G; \mathbb{Z})$ for some subgroups $G \subset \Sigma_q$, e.g., for $q = p^k$, $k \geq 2$, 2.6.3 and 2.8.2 show respectively that this restriction is zero for $G = \mathbb{Z}/p^k$ but nonzero for $G = (\mathbb{Z}/p^k)$. Could it not be that for a $q \neq p^k$ this class is nonzero despite the fact 2.9.1 that its restriction to all simply transitive subgroups G is zero? If so the "continuous" Tverberg would extend to such a q, because we obviously have a continuous Σ_q-map from the free and N-dimensional Σ_q-complex $E_N \Sigma_q$ to the $(N-1)$-connected Σ_q-complex $Q \cdot \ldots \cdot Q$. Unfortunately the answer to this new question is also "no".

2.9.3.

Theorem 5. *The Euler class of the Σ_q-representation $\mathbb{L}^{\perp}$ is nonzero iff q is a prime power.*

Proof. By 2.8.2 it only remains to look at the case $q \neq p^k$. One has $H^N(\Sigma_q; \mathbb{Z}) = \oplus_p H^N(\Sigma_q; \mathbb{Z}, p)$, where p runs over all primes, and $H^N(\Sigma_q; \mathbb{Z}, p)$ denotes the p-primary component of $H^N(\Sigma_q; \mathbb{Z})$. If $P \subset \Sigma_q$ is a p-Sylow subgroup then — see Cartan-Eilenberg [**7**, p. 259, Thm. 10.1] —

restriction gives us a monomorphism $H^N(\Sigma_q;\mathbb{Z},p) \to H^N(P;\mathbb{Z})$. So it suffices to show that the restriction of our class to each $H^N(P;\mathbb{Z})$ is zero. To see this note that $|P|$ is not divisible by $q \neq p^k$, so P does not act transitively on Q, so there are trivial P-representations outside the diagonal of $\mathbb{L}$, i.e., in $\mathbb{L}^\perp$. □

Note that, the Σ_q-action on $E_N(Q)$ being not free, this still leaves open the question whether, for $q \neq p^k$, one can have a continuous Σ_q-map $E_N(Q) \to \mathbb{L}^\perp$ having no zeros? It seems that $U(q)$-actions are called for to settle this point, so we postpone it to a sequel which will deal with infinite group actions.

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