The prettiest composition, part three
(translation of ਸਭ ਤੋਂ ਪਿਆਰੀ ਰੱਚਣਾ, ਭਾਗ ਤੀਜਾ)

34. The method of note 33, giving real roots of any real degree n equation in one unknown $\mathtt{x}$, gives rise to many questions. Can it, as seems likely, be pushed to get also the conjugate pairs of complex roots, and its geometry complexified to solve as well equations over $\mathbb{C}$? More tantalising is whether there is an n-spherical geometric version which solves all real homogenous degree n equations in two unknowns $\mathtt{x}$ and $\mathtt{y}$ and is distortion-free ? If so maybe even an n-spherical Vieta's method solving all equations in the closed projective real n-swallowtail, i.e., all equations of $\mathbb{R}P^n$ with all roots real, an n-manifold-with-boundary whose topology we have worked out in notes (ਬ|੨੧,੨੨) of aft FTA ? So by analytic continuation—the complex projective n-swallowtail, i.e., all equations of $\mathbb{C}P^n$ with n distinct roots, being connected and dense—a function theoretic method solving all equations over $\mathbb{C}$? However, for us it is the real n-swallowtail itself (and more generally the fundamental partition of $\mathbb{R}P^n$ or its cover S^n) that is natural, not explicit formulas for solving equations in it, these should flow out anyway once we have worked out its geometry as well.

35. The above path may indeed take us, from Khayyam's method to what Jordan had called, *résolution par les équations de la bissection des fonctions hyperelliptiques* ? For, this method in his book, *Traité des Substitutions* (1870), page 380, is the 'function theoretic method solving all equations over $\mathbb{C}$' that we too shall end up with ? (For explicit formulas using the many special functions tied to this method see Umemura, *Resolution of algebraic equations by theta constants* (1984), and its references.) To each point in the complex n-swallowtail is attached a hyperelliptic curve, a closed 2-manifold of genus $[\frac{n-1}{2}]$, and these hyperelliptic functions uniformize it. Despite the fact that the genus is two for $n = 5$, quintics had been solved before by Hermite and Kronecker in 1858 using only elliptic modular functions. On this very special case $n = 5$, light was really shed only later by a book of Klein on the icosahedron. Likewise, the general notion of uniformizing, and so 'solving' any complex projective curve by his automorphic functions, was also established later by Poincaré.

36. The curves $P(\alpha), \alpha \in \mathbb{R}$ of note 33 pass through $p \in \mathbb{R}^{n-1}$ as many times as its distinct real roots, and at most once through any other point:- For p is its own mirror image iff $p \in$@, and its images in other @'s are distinct. $\square$

If n is odd, and even if p has a real root, then P is not in a proper flat of $\mathbb{R}^{n-1}$:- If $c_0(p_0 + 2t(\alpha)) + c_1(p_1 + \alpha 2t(\alpha)) + \cdots + c_{n-2}(p_{n-2} + \alpha^{n-2}2t(\alpha))$, where $t(\alpha) = -\frac{\alpha^n + p_{n-2}\alpha^{n-2} + \cdots + p_1\alpha + p_0}{\alpha^{2(n-2)} + \cdots + \alpha^2 + 1}$, is constant for all α, then putting $\alpha = 0$ we see that it is identically $-c_0p_0 + c_1p_1 + \cdots + c_{n-2}p_{n-2}$. So $c_0 2t(\alpha) + c_1\alpha 2t(\alpha) + \cdots + c_{n-2}\alpha^{n-2}2t(\alpha) = -2c_0p_0$, that is, $(c_0 + c_1\alpha + \cdots + c_{n-2}\alpha^{n-2})(\alpha^n + p_{n-2}\alpha^{n-2} + \cdots + p_1\alpha + p_0) = c_0p_0(\alpha^{2(n-2)} + \cdots + \alpha^2 + 1)$. So $c_{n-2} = c_{n-3} = 0$, $c_{n-4} = c_0p_0$ and $c_{n-5} = 0$. If $c_0p_0 = 0$ all c_i are zero, if not dividing by $c_0p_0 \neq 0$ gives $(\alpha^{n-4} + b_{n-6}\alpha^{n-6} + \cdots + b_0)(\alpha^n + p_{n-2}\alpha^{n-2} + \cdots + p_1\alpha + p_0) = \alpha^{2(n-2)} + \cdots + \alpha^2 + 1$ where $b_i = c_i/c_op_0$, but this can't be if the second factor is zero for some α, because the right side is positive for all α. $\square$

For n even, there are finitely many exceptions p for which the above result is not true, viz., any p with roots $n/2$ distinct pairs of complex conjugate $(2n-2)$th roots of unity :- Use factorization above noting $\alpha^{2(n-2)} + \cdots + \alpha^2 + 1$ has as roots all $(2n-2)$th roots of unity other than ± 1. □ For any such exceptional or special point p, the codimension of $\mathrm{aff}(P) \subset \mathbb{R}^{n-1}$ is one:- because the remaining $n-2-n/2$ pairs fix the first factor and so c_i/c_0. □

So, the number of special points p is at most $\binom{n-2}{n/2}$ but usually smaller, for, not all choices of $n/2$ pairs have sum zero, while we are constrained to equations with sum of roots zero, for example, there is no exceptional p for $n=6$:- the 10th roots of unity are equally spaced at angles $\pi/5$ on the unit circle, of the 4 complex conjugate pairs of these no 3 have sum zero, because then the sum $w+\overline{w}$ of the last would be zero, which is false. □ On the other hand, there are at least $\binom{n/2-1}{n/4}$ special p for any n divisible by 4 :- each quartet $\{\pm w, \pm\overline{w}\}$ of $(2n-2)$th complex roots of unity has sum zero, so any $n/4$ quartets give an exception. □ But, $4|n$ is not necessary, there are exceptions for $n=10$:- the 18th roots of unity are equally spaced at angles $\pi/9$ on the unit circle, take any 3 complex ones equally spaced at angles $2\pi/3$ together with their conjugates, the remaining 5 conjugate pairs give an exceptional p. □

37. That circular cuts on one special parabola solve not only degree 3 but also degree 4 equations – note 24 – may have been known to Khayyam, but was pointed out only by Hachtroudi. This parabola is tied to the one special point for $n=4$:- Namely $p=(1,0,1)$, when $t(\alpha)=-1$ and $P(\alpha)=(-1,-2\alpha,1-2\alpha^2)$, i.e., parabola $u_0=-1, u_2=1-u_1^2/2$. Therefore, the real roots of any $\mathtt{x}^4+u_2\mathtt{x}^2+u_1\mathtt{x}+u_0=0$ can be read from the cuts made on this parabola by the 2-sphere with centre (u_0,u_1,u_2) through $(1,0,1)$, i.e., the cuts made on it by the circle on this sphere lying on the plane $u_0=-1$, i.e., the circle with centre $(-1,u_1,u_2)$ and radius $\sqrt{-4u_0+u_1^2+(u_2-1)^2}$ (if this is imaginary there is no real root). So any cubic $\mathtt{x}^3+u_2\mathtt{x}+u_1=0$ can be solved by the cuts on the parabola $u_2=1-u_1^2/2$ in the plane $u_0=0$ made by the circle with centre (u_1,u_2) and radius $\sqrt{u_1^2+(u_2-1)^2}$, i.e., through $(0,1)$.□

For $n>4$ circles won't do, we need $(n-2)$-, or at best $(n-3)$-spherical cuts on these finitely many special 'generalized parabolas', any n points of $P(\alpha)$ with sum of parameters zero are now on an $(n-3)$-sphere :- Namely the intersection of $\mathrm{aff}(P)$ and the $(n-2)$-sphere through p with centre $\odot$ the degree n equation having these n parameter values as roots. □

Projectively compactifying all real degree n equations with n distinct real roots gives the n-swallowtail of note 34 above. For n odd it is circle times closed $(n-1)$-ball, but for n even their $\mathbb{Z}/2$-twisted product. That they too occur only for n even, and only finitely often, suggests that, special generalized parabolas are tied to the mod 2 invariants of van Kampen and Stiefel ? However, if we complexify and projectively compactify the same space we get $\mathbb{C}P^n$ and the integer invariants of Pontryagin for dimensions time four; the degrees of the two polynomial factors in note 36 also differ by four ...

38. Without 'sum of the roots zero' the method of note 33 becomes this:- Any equation $\mathtt{x}^n+u_{n-1}\mathtt{x}^{n-1}+\cdots+u_0=0$ is a point $(u_0,\ldots,u_{n-1})=\odot$ of $\mathbb{R}^n$,

set @ of all having root α its hyperplane $\alpha^n + u_{n-1}\alpha^{n-1} + \cdots + u_0 = 0$, mirror images of a fixed $p = (p_0, ..., p_{n-1})$ in these form its *khayyam curve* $P(\alpha) = (p_0+2t(\alpha), p_1+\alpha 2t(\alpha), \ldots, p_{n-1}+\alpha^{n-1}2t(\alpha))$, $t(\alpha) = -\frac{\alpha^n+p_{n-1}\alpha^{n-1}+\cdots+p_1\alpha+p_0}{\alpha^{2(n-1)}+\cdots+\alpha^2+1}$; so, the real roots of any degree n equation $\odot$ are given by the cuts made on the fixed curve P of $\mathbb{R}^n$ by the $(n-1)$-sphere with centre $\odot$ through p. $\square$

Further, $\mathrm{aff}(P) \neq \mathbb{R}^n$ iff $\mathtt{x}^n+p_{n-1}\mathtt{x}^{n-1}+\cdots+p_0$ divides $\mathtt{x}^{2(n-1)}+\cdots+\mathtt{x}^2+1$; for these–now exactly–$\binom{n-1}{n/2}$ special points, $\mathrm{aff}(P)$ has codimension one and does not contain p :- argue as in note 36. $\square$ Notably, $p = (1, 0, \ldots, 0)$ is special for all n even, and its khayyam curve P lies in $u_0 + u_2 + \cdots + u_{n-2} = -1$:- because $1 - 2\frac{\alpha^n+1}{\alpha^{2(n-1)}+\cdots+\alpha^2+1}(1 + \alpha^2 + \cdots + \alpha^{n-2}) = 1 - 2 = -1$. $\square$

But alas! to solve cubics we now must use space curves P. To solve degree 4 equations if we use one of the three special points $p = (1, 0, 0, 0), (1, \pm\sqrt{2}, 2, \pm\sqrt{2})$ of $\mathbb{R}^4$ corresponding to the three divisors $\mathtt{x}^4 + 1, \mathtt{x}^4 \pm \sqrt{2}\mathtt{x}^3 + 2\mathtt{x}^2 \pm \sqrt{2}\mathtt{x} + 1$ of $\mathtt{x}^6 + \mathtt{x}^4 + \mathtt{x}^2 + 1$, their curves P, being in 3-dimensional flats, are also only space curves. And yay! for $n = 2$ the unique special point p gives us this circle method for solving any quadratic $\mathtt{x}^2 + u_1\mathtt{x} + u_0 = 0$:- draw in $\mathbb{R}^2$ the circle with centre (u_0, u_1) through $p = (1, 0)$ and note its ≤ 2 cuts 'α' on the line $P(\alpha) = (-1, -2\alpha)$. $\square$ But let us not forget! we have also that school method of solving quadratics by – see picture – 'completing a square'.

39. We can use too the depicted completing a cube, or even completing an

n-cube. With sweat we had solved half the cubics thus, but it seems for $n > 3$ we will not get much this way, beyond say getting rid of the second term. In fact, the party has just begun! this method from school is the key to understanding the basic symmetries of the space of all equations.

For example, euclidean motions $x \mapsto \pm x + t$ of $\mathbb{R}$ act on the space of equations in $\mathtt{x}$ by substitutions $\mathtt{x} \mapsto \pm\mathtt{x} + t$, and the orbits of their action can be computed using $(a+b)^n = \sum \binom{n}{i} a^i b^{n-i}$:- i.e., the natal relationship between addition and multiplication that Baby Algebra is proving in the picture. $\square$

The fundamental strata of $\mathbb{R}^n$ get partitioned further into the orbits of this action :- For, if an equation has exactly $n - 2k$ distinct real roots, so too have all the equations obtained by making these substitutions. $\square$

However, the stratum 'all roots equal' of $\mathbb{R}^n$ has just one orbit, and this *cuspidal curve* is, up to rescaling, a moment curve :- For $\mathtt{x}^n + u_{n-1}\mathtt{x}^{n-1} + \cdots + u_1\mathtt{x} + u_0 = (\mathtt{x} - t)^n$ gives $u_i(t) = (-1)^{n-i}\binom{n}{i} t^{n-i}, 0 \le i \le n-1, t \in \mathbb{R}$. $\square$

The hyperplane of $\mathbb{R}^n$ which kisses this curve at its point 'α' is precisely the hyperplane @ :- The intersections of $\alpha^n + u_{n-1}\alpha^{n-1} + \cdots + u_0 = 0$ with the curve are given by putting $u_i = (-1)^{n-i}\binom{n}{i} t^{n-i}$, this gives $(\alpha - t)^n = 0$, and solving for t, so @ cuts the curve n times at $t = \alpha$. $\square$

So, the cuspidal curve determines the fundamental parts of $\mathbb{R}^n$ and the graph G above it of all real roots of all $\odot \in \mathbb{R}^n$:- $G \subset \mathbb{R}^n \times \mathbb{R}$ is the disjoint union of the flats at time $t \in \mathbb{R}$ parallel to the osculating hyperplanes of this curve at its points 't', and the maximal *open* subsets on which the fibers of $G \to \mathbb{R}^n$ have cardinality n, $n-2$, etc., are the fundamental parts of $\mathbb{R}^n$. $\square$

That, the real roots α of any $\odot \in \mathbb{R}^n$ can be read from the points 'α' of the cuspidal curve with osculating hyperplanes passing through it, is stronger than the 'weak generalization of Khayyam's method' given in note 28 :- For, if $\alpha \neq 0$ then @$\cap\mathbb{R}^{n-1}$ kisses the 'red curves' at their 'α'. $\square$

So discarding 'sum of roots zero' has resulted in our using, instead of those two 'red curves' missing a common sharp singularity, the single smooth curve formed by all these cusps. Further this cuspidal curve, more generally, any orbit is generated by the involutions $\mathtt{x} \mapsto -\mathtt{x} + 2\alpha$:- because the motions of the real line are generated by its reflections $t \mapsto -t + 2\alpha$. $\square$ These involutions $\tilde{\alpha}$ preserve the hyperplanes @ :- Indeed, t is a root of $\odot$ iff $-t + 2\alpha$ is a root of $\tilde{\alpha}(\odot)$. $\square$ However, these order 2 nonlinear maps of $\mathbb{R}^n$ are not reflections, and for n even, the fixed points of $\tilde{\alpha}$ are not even in @ :- For, an equation is fixed iff its roots are symmetric around α. $\square$ So, even if there is some distance on $\mathbb{R}^n$ invariant under all involutions, it does not appear that it will be of much use for this osculating hyperplanes method of solving equations.

40. The addition and multiplication of segments are no different from their discrete forms – which even a baby playing with blocks can grasp – if we are okay with 30.199... = 30.2, see PG&R-V note 30.199... This had stirred in us the memory of that natal bond between the two which we are visualizing as the above steady motion in the space of real degree n equations. But what magic is this now : from just *one* (the cuspidal) orbit of this *steady* motion, are born *all* the manifolds of the fundamental partition ! Which naturally makes us wonder

:- is this tied to the cartesian creation of PG&R ? (We recall and emphasize that the 'cartesian motions' used in it were very *unsteady*, and there is *no* direct physical relation between some such cartesian motions that we'll meet below in notes 42, 43, etc., and this steady binomial or baby motion). Mulling on this, let us however own up that, of the n-manifolds of the partition, *per se*, only the n-swallowtail —all real degree n equations with n distinct real roots—was created from the orbit, for the others we implicitly used complex numbers :- the kissing hyperplanes only give real roots, which does not fix the stratum of an $\odot$, for example, a degree 4 equation with distinct real roots r and s may have roots $\{r, r, s, s\}$, etc., or $\{r, s, a \pm ib\}$, but yes, only the last possibility is in a maximal *open* stratum. $\square$ So the question to ask is this :- is this creation of the swallowtail from one full orbit tied to cartesian creation, and if so, does the orbit create as well in finite times closed submanifolds of the same, which evolve in it like the ones in note 28 of PG&R-V ?

41. Complex numbers are much too rich, distractingly rich. To avoid these distractions we shall focus on the real swallowtail, or at the most its closure, for only this closed subset of $\mathbb{R}P^n \subset \mathbb{C}P^n$ seems natural (but it seems likely that complex – maybe even quaternionic and octonionic – multiplication will arise by itself from this baby action only). Nevertheless, we note that

Note 39 complexifies to solve all equations :- Any n points of the cuspidal curve determine the equation $\odot$ of the swallowtail having the corresponding n roots, and its kissing hyperplanes through $\odot$ give back these points. The closure of the cuspidal curve has just one extra point ∞, the homogenous equation in $\mathtt{x}$ and $\mathtt{y}$ with all roots at infinity, so the closed swallowtail—all real homogenous equations in $\mathtt{x}$ and $\mathtt{y}$ with all roots real—is homeomorphic to the nth symmetric power of S^1. By the FTA its complexification is $\mathbb{C}P^n$, while that of the closed cuspidal curve is $S^2 = \mathbb{C} \cup \infty$, whose kissing complex hyperplanes through any $\odot \in \mathbb{C}P^n$ solve this equation, giving us the inverse fundamental homeomorphism from $\mathbb{C}P^n$ to the nth symmetric power $\mathrm{Sym}^n(S^2)$. $\square$

So we are back at the exact same spot, viz., PG&R-V footnote 20, from which this burgeoning offshoot of PG&R had once sprouted. Following this, we had shown in FTA that the *'magic of complex numbers'* lies in the fact that their dimension $m = 2$ is the only one such that the symmetric powers $\mathrm{Sym}^n(M^m)$ of m-manifolds M^m are also manifolds. However the dimension $m = 1$ of the real numbers is *almost* as good, these symmetric powers are now manifolds-with-boundary. A very big plus now is that there is *only one* closed manifold S^1, and in the next paper aft FTA, amongst many other things, we had computed all $\mathrm{Sym}^n(S^1)$. As against this, in dimension 2 there is *an infinite stable* of closed manifolds M^2—an example of *'distractingly rich'*—whose symmetric powers, though manifolds, are still not fully understood. In FTA we had also computed $\mathrm{Sym}^n(\mathbb{R}^m), \forall\, m, n$; maybe for $m = 4$ and $m = 8$ their top strata are quaternionic and octonionic swallowtails in some sense ?

42. Responding to the query ending note 40, any cartesian motion in which only one real root of an equation can change at any time, and does change at some time without losing its identity, creates the swallowtail :- Each equation $\odot$

of the swallowtail is the intersection $\alpha_1 \cap \ldots \cap \alpha_n$ of the n osculating hyperplanes solving it. Omitting one of them at a time gives us n lines $\alpha_1 \cap \ldots \cap \hat{\alpha_j} \cap \ldots \cap \alpha_n$ through each point $\odot$ of the swallowtail, which remain in the swallowtail but for $n-1$ points, at which the varying root loses its identity, and coincides with one of the fixed roots $\alpha_i, i \neq j$. So, minus these points of tangency with the top strata of the boundary, each line has n parts in the swallowtail. When any $\odot$ moves it stays in its part in one of these n lines through it; also we know that each point of the swallowtail moves sooner or later in each of these n lines. So, the *minimal set*—a minimal saturated union of orbits of the cartesian motion—containing one $\odot$ with n distinct real roots is the union of all the parts of all these lines, i.e., it is the entire swallowtail. □

43. Such a motion creates also all boundary strata of the swallowtail :- The 'identity' of a distinct root is its distinctness and more generally of any root is its multiplicity. The stratum of any $\odot$ with all roots real is determined by its *type,* i.e., the sequence of multiplicities from that of its least to biggest root. Only one root can vary preserving multiplicity defines maximal curved open intervals in each stratum; as many through each point as the number of distinct roots; also sooner or later each point does move on each of its curved intervals; which implies, the minimal set of $\odot$ is its entire stratum. For, though one orbit—the image under $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ of a flow line—of the cartesian motion, can be much smaller than a curved interval, sooner or later any stagnant point does move a bit in each of its intervals, therefore the minimal saturated union of orbits containing the point $\odot$ is its entire stratum. □

44. What about the minimal set of an equation $\odot \in \mathbb{R}^n$ having some complex conjugate roots? Since only real roots move retaining identity, we'll now only get that leaf of a certain foliation of the stratum in which this equation $\odot$ lies. At one extreme of no real root, which can happen only if n is even, its point foliation; at the other, of all roots real, its one-leaf foliation.

Why not complexify to create likewise the connected manifold strata of $\mathbb{C}^n$, which consist now of all complex degree n equations having multiplicities of given frequencies ? No problem, but it seems wiser to not admit complex time across that lakshman rekha; otherwise, for periodicity for one, we would need to consort with that entire 'infinite stable' of closed 2-manifolds.

We'll stay loyal to real one dimensional time, and usual S^1-periodicity. This suffices—see PG&R note 28—for the genesis and evolution of all sorts of closed manifolds in the swallowtail as periodic perceptions of a cartesian motion.

We used the total order of the coordinates of $\mathbb{R}^n$, but we have made *no* use of euclidean distance; in fact, its cartesian creation is pointing towards another and invariant under involutions distance for each stratum of the closed swallowtail, akin to the cayley distance of PG&R ...

45. In note 42 above we had written our old pal @ as just α since there was no danger of confusion. Now we'll write it instead as ${}^1\alpha$, and more generally, the subset of all $\odot \in \mathbb{R}^n$ having $\alpha \in \mathbb{R}$ as a root of multiplicity $\geq m$ will be denoted by ${}^m\alpha$; this is a flat of codimension m :- For $(u_0, ..., u_{n-1}) \in {}^m\alpha$ iff $\frac{d^j}{d\alpha^j}(u_0 + u_1\alpha + \cdots + u_{n-1}\alpha^{n-1} + \alpha^n) = 0$ for all $0 \leq j < m$, and the leading

$m \times m$ triangular matrix of coefficients of these m linear equations has nonzero diagonal terms $j!$, $0 \leq j < m$. $\square$

This is equivalent to saying that, ${}^m\alpha$ is the codimension m flat of $\mathbb{R}^n$ which kisses the cuspidal curve at 'α':- For this curve cuts it $n - m + 1$ times in 'α'. $\square$ In particular, ${}^{n-1}\alpha$ is the tangent line to the cuspidal curve at 'α' and the 0-dimensional flat ${}^n\alpha$ is this cusp 'α' itself.

The affine spans of the 'maximal curved open intervals' of note 43 have dimensions equal to the multiplicity of the varying root :- If a point $\odot$ of the closed swallowtail has in ascending order d distinct roots α_j having multiplicities m_j, then its stratum has the type $m_1 \ldots m_j \ldots m_d$, and we have $\odot = {}^{m_1}\alpha_1 \cap \ldots \cap {}^{m_j}\alpha_j \cap \ldots \cap {}^{m_d}\alpha_d$. Omitting the jth flat from this intersection gives the m_j dimensional affine span of the moment-like curve through $\odot$ obtained by varying only the jth root keeping its multiplicity m_j intact. $\square$

Therefore, if a simple root is varying the curved intervals are in fact straight, while, the most curved of them all is the cuspidal curve, which is also the only one which is doubly infinite :- When its multiplicity is less than n, the varying root is bounded below or above by a fixed root. $\square$

46. Was Descartes—whose not so well known dictum that shape and matter are but forms of motion we have made our own—led to his well known coordinate axes by a similar chain of thought? Dunno!

Anyway, through each $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ go the n lines along which only one coordinate varies and, all of $\mathbb{R}^n$ is the unique minimal set of any cartesian motion of itself such that, each point can move only on one of these lines through it, and does move sooner or later on each of them.

From this perspective, the euclidean distance is inapt for $n \geq 2$, it does not give importance to these n directions. The cartesian–meaning natural–distance now is the least that we need to walk to go from $\mathbf{x}$ and $\mathbf{y}$ in any such way, i.e., $|x_1 - y_1| + \cdots + |x_n - y_n|$. It gives us the same topology as the euclidean distance, but the geometry of this n-space is different, e.g., there are infinitely many shortest paths between most pairs of points.

Likewise, any open connected set $U \subset \mathbb{R}^n$ is the unique minimal set of any cartesian motion of itself of the above kind. For, U being open and connected, we can go from $\mathbf{x} \in U$ to $\mathbf{y} \in U$, staying within U, and walking always parallel to some axis. Since the permitted paths are now lesser, the infimum that we need to walk may be more : the cartesian distance of U tied to its creation may not be the restriction of the cartesian distance of $\mathbb{R}^n$.

This infimum is more precisely the nonrelativistic cartesian distance of U tied to this creation. Each segment we travel is not necessarily on a line entirely in U. If not, then we should perhaps use cayley distance to measure this stretch, which depends on the one or two points at which the extended segment first exits U. After this modification the infimum over all permitted paths gives the relativistic cartesian distance of U tied to this genesis.

We recall that, up to homotopy type any closed manifold is such a U; and for many things, for example, de Rham and cyclic cohomology, we can use instead of the manifold this $U \subset \mathbb{R}^n$, which often makes these ideas far clearer.

47. Between the last note and what was going on before it, the difference is that then the total order of the coordinates was also in play, which had made that game a tad noncommutative. It was by using this that any $\odot \in \mathbb{R}^n$ was made a degree n equation in $\mathtt{x}$. Mulling over their roots then, we had taken to gazing at the fundamental partition of $\mathbb{R}^n$, especially the n-swallowtail. Its cartesian genesis in note 42 is only superficially like the one of the last note. Because we don't know all about the roots, it is a bit hidden or quantized. On the plus side, there is a very pleasant feature in it which is not there in the classical picture of the lasr note. This picture is not static, it is moving ! Under that steady baby flow, which is but another name for the fundamental relation between addition and multiplication proved by Baby Algebra.

The distances analogous to note 46 on the swallowtail tied to its creation in note 42 are preserved by its involutions and so by its baby flow :- The constraint that the moving root α_j remains simple imprisons it between α_{j-1} and α_{j+1}. Between any two numbers $a < b$ in this jail, the nonrelativistic distance is $b - a$ and the relativistic $\log\left(\frac{b-\alpha_{j-1}}{a-\alpha_{j-1}} \times \frac{\alpha_{j+1}-a}{\alpha_{j+1}-b}\right)$. The second formula makes sense even if one of the jailers is at infinity. The multiplicative triangle inequality of this cross ratio was known even to Pappus – PG&R – the log of Cayley made it additive. Under any involution of note 39 the roots of all $\odot$ get reflected in some fixed number. So only the signs of the differences used in this recipe change, this distance is invariant under the involution. $\square$

This geometry is very flexible ! The swallowtail is isometric with the open set U of strictly increasing n-tuples $(\alpha_1, \ldots, \alpha_n)$ of $\mathbb{R}^n$ equipped with the cartesian distances of note 46. Flexible to the extent that those n parts of note 42 all on the same straight line are now parallel to the respective axes, the ones parallel to the first and last being infinite. As against this, the moment-like cuspidal curve is now dead straight, so with no well-defined osculating planes that we can play with. On the other hand, U is convex, so its nonrelativistic cartesian distance is the restriction of the distance $|x_1 - y_1| + \cdots + |x_n - y_n|$ of $\mathbb{R}^n$, and to examine the relativistic distance too this is the right picture. By the way, this set U is the interior of the chamber $W = W^{(1)}$ of FTA whose generalization $W^{(m)} \subset \mathbb{R}^{mn}$ had figured in our analysis of $\mathrm{Sym}^n(\mathbb{R}^m)$.

But we have seen only one side of the *full baby flow* or action so far! It will reveal itself fully once we understand projective compactification in a plain and natural way. This is the agenda of the next note.

48. Our interest is in equations $\cdots + a_j \mathtt{x}^j + \cdots + a_0 = 0$ in $\mathtt{x}$, where, at least one, and at most finitely many, of the coefficients $a_j \in \mathbb{R}$ are nonzero. Since multiplication by a nonzero number does not change an equation, the space of all equations in $\mathtt{x}$ is $\mathbb{R}P^\infty$, the space of equivalence classes $[\ldots, a_j, \ldots, a_0]$ of such sequences upto multiples. The degree of an equation is the biggest j such that a_j is nonzero. All equations in $\mathtt{x}$ of degree $\leq n$ form a real projective n-space $\mathbb{R}P^n$, the subspace of $\mathbb{R}P^\infty$ containing all $[\ldots, 0, a_n, \ldots, a_j, \ldots, a_0]$; and, all equations in $\mathtt{x}$ of degree exactly n form an $\mathbb{R}^n$, the subspace of $\mathbb{R}P^n$ containing all $[\ldots, 0, a_n, \ldots, a_j, \ldots, a_0]$ with $a_n \neq 0$, making this coefficient $a_n = 1$ gives a bijective correspondence with all ordered n-tuples $(a_0, \ldots a_{n-1})$.

Homogenization is prompted by: an equation $\odot \in \mathbb{R}P^n$ has a lower degree $j < n$ iff it has infinity as a root of multiplicity $n-j$:- The equations of degree one or zero form $\mathbb{R}P^1 = \mathbb{R}^1 \cup \mathbb{R}^0$, where point $\mathbb{R}^0$ is the unique degree zero equation $1 = 0$, which has no finite root $x \in \mathbb{R}$, but we'll say it has the infinite root $x = \infty$ in the extended reals $\mathbb{R} \cup \infty = S^1$. More precisely, we deem the extended reals also as equivalence classes $[x, y]$ of pairs of reals not both zero—so $x \in \mathbb{R}$ is $[x, 1]$ and $\infty = [1, 0]$—and this will be a root of any equation $a_1\mathtt{x}+a_0 = 0$ of degree ≤ 1 iff $a_1x + a_0y = 0$. Likewise, all equations $a_n\mathtt{x}^n + \cdots + a_0 = 0$ of degree $\leq n$ form $\mathbb{R}P^n = \mathbb{R}^n \cup \ldots \cup \mathbb{R}^j \cup \ldots \cup \mathbb{R}^0$—a decomposition of projective n-space into cells, one of each dimension—and an extended real $[x, y]$ will be a root iff $\mathtt{x} = x, \mathtt{y} = y$ satisfy the degree n homogenous equation $a_n\mathtt{x}^n + \cdots + a_j\mathtt{x}^j\mathtt{y}^{n-j} + \cdots + a_0\mathtt{y}^n = 0$ in $\mathtt{x}$ and $\mathtt{y}$. Equivalently iff its left side is divisible by $y\mathtt{x}-x\mathtt{y}$. The cell $\mathbb{R}^j$ consists of all equations of degree j, i.e, $a_n = \cdots = a_{j+1} = 0, a_j \neq 0$, i.e., $\mathtt{y}^{n-j}$ factors, i.e., $[\mathtt{x}, \mathtt{y}] = [1, 0] = \infty$ is a root of multiplicity $n - j$. $\square$

Homogenization is defined only for equations with degree at most some n, but then, there is no reason to treat $\mathtt{y}$ in a stepmotherly way, indeed this baby gives symmetries poincaré dual to those of note 39 :- The extended reals has two charts $\mathbb{R}$, with any nonzero number x of the first glued to its reciprocal y in the second, and the zero of each deemed the infinity of the other. The space of all degree n homogenous equations $a_n\mathtt{x}^n+\cdots+a_j\mathtt{x}^j\mathtt{y}^{n-j}+\cdots+a_0\mathtt{y}^n = 0$ in $\mathtt{x}$ and $\mathtt{y}$, viz., $\mathbb{R}P^n$ is equally the space of all equations $a_n+\cdots+a_j\mathtt{y}^{n-j}+\cdots+a_0\mathtt{y}^n = 0$ in $\mathtt{y}$ of degree $\leq n$. Note this degree is equal to $n-j$ iff j is least such that $a_j \neq 0$. The poincaré dual cell decomposition $\mathbb{R}P^n = {}^*\mathbb{R}^0 \cup \ldots \cup {}^*\mathbb{R}^{n-j} \cup \ldots \cup {}^*\mathbb{R}^n$ has as its $(n - j)$-cell ${}^*\mathbb{R}^{n-j}$ all equations $a_n + \cdots + a_j\mathtt{y}^{n-j} = 0, a_j \neq 0$ in $\mathtt{y}$ of degree $n - j$, that is, all having $\mathtt{y} = \infty$, i.e., $[0, 1]$, as a root of multiplicity j. The reflections of the second chart of the extended reals also act on $\mathbb{R}P^n$ by the involutions $\mathtt{y} \mapsto s-\mathtt{y}$. This action preserves degree with respect to $\mathtt{y}$, that is, the dual cell decomposition of $\mathbb{R}P^n$. $\square$ This action too, indeed anything involving that relation between addition and multiplication proved by baby Algebra, is quite naturally called algebra-ic by one and all!

More generally, any linear isomorphism of the extended reals acts on $\mathbb{R}P^n$ by substitutions, preserving its fundamental partition, but not necessarily either cell subdivision:- The extended reals $[x, y]$ as lines of $\mathbb{R}^2$ through its origin admit these bijections $[x, y] \mapsto [ux + ty, sx + vy], uv - st \neq 0$, and the corresponding substitutions $\mathtt{x} \mapsto u\mathtt{x} + t\mathtt{y}, \mathtt{y} \mapsto s\mathtt{x} + v\mathtt{y}$ in all degree n homogenous equations in $\mathtt{x}$ and $\mathtt{y}$ give bijections of $\mathbb{R}P^n$. Since an extended real $[x, y]$ is a root of the new equation iff $[ux+ty, sx+vy]$ is a root of the old with the same multiplicity, they preserves the fundamental partition of $\mathbb{R}P^n$. $\square$ This because its $[\frac{n}{2}] + 1$ disjoint parts consist of all equations of $\mathbb{R}P^n$ having the same number counted with multiplicity of extended real roots. For the kth part this number is $n-2k$, the remaining roots being k conjugate pairs of complex numbers in $\mathbb{C} \setminus \mathbb{R}$, the complement of the circle $\mathbb{R} \cup \infty$ of extended reals in the riemann sphere $\mathbb{C} \cup \infty$. We recall also that, the interior of any part is connected and consists of its equations with *simple* extended real roots, but the conjugate pairs of complex roots *can* repeat—see aft FTA—and our interest is mainly in the part $k = 0$, i.e., the projective n-swallowtail.

The linear isomorphisms of the extended reals $\mathbb{R}P^1$ (or any $\mathbb{R}P^n$) do not depend on the coordinates, these only give their matrices $\begin{bmatrix} u & t \\ s & v \end{bmatrix} \in GL(2,\mathbb{R})$. Since the do-nothing matrices are diagonal with entries same, the orientation preserving isomorphisms are in bijective correspondence with pairs of matrices $\pm A \in SL(2,\mathbb{R})$, and the orientation reversing with pairs of determinant -1. (Likewise for n odd, but for n even the do-nothing matrices $\pm I$ are in different components, $SL(n+1,\mathbb{R})$ gives all isomorphisms of the non-orientable $\mathbb{R}P^n$.) The matrices of $SL(2,\mathbb{R})$ preserve the area of $\mathbb{R}^2$, those preserving distance form a circular deformation retract $SO(2,\mathbb{R}) = S^1$.

Once again, the total order of its coordinates identifies any $\odot \in \mathbb{R}^n$ with an equation in $\mathtt{x}$ of degree $= n$; but it is wiser to consider equations in $\mathtt{x}$ of all (for we shouldn't just throw away what we may have learnt about lower) degrees $\leq n$; their space $\mathbb{R}P^n$ is nicer, a closed n-manifold; and we see the need of a root $x = \infty$; homogenization and the dual unknown $\mathtt{y}$ emerge ... our total order, much like a morse function, subdivides $\mathbb{R}P^n$ into cells and dual cells; preserved respectively by $\mathtt{x} \mapsto u\mathtt{x} + t\mathtt{y}, \mathtt{y} \mapsto v\mathtt{y}$ and $\mathtt{x} \mapsto u\mathtt{x}, \mathtt{y} \mapsto s\mathtt{x} + v\mathtt{y}$, but neither cell subdivision is preserved by the action of all matrices $\begin{bmatrix} u & t \\ s & v \end{bmatrix} \in SL(2,\mathbb{R})$ on $\mathbb{R}P^n$ ... but same total order also gives a 'round subdivision', the fundamental partition with kth part all equations $\odot \in \mathbb{R}P^n$ with $n-2k$ extended real roots counted with multiplicity; which *is* preserved by all these subsititutions; and as we'll soon see it is indeed quite 'round' ... for example, if all roots are finite and real this action rotates their increasing sequence through infinity, which reminds us that a rotation of cubics with sum of roots zero had given Vieta's method : does this baby action give something similar for all n?

49. Rotating an equation $\odot \in \mathbb{R}P^n$, having only extended real roots $[x,y]$, by $\pm \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ to $\theta(\odot)$ is equivalent to rotating its roots by the inverse matrix to $[x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta], 0 \leq \theta < \pi$. In particular, if the roots of $\odot$ are finite $[x_i, 1]$ and simple, $x_1 < \cdots < x_n$, the n nonzero rotations which successively make them infinity or $[1,0]$ are given by $x_i = -\cot\theta_i$, i.e., the n rotated equations whose degree in $\mathtt{x}$ is $n-1$ are $\theta_i(\odot)$. Before working out how the remaining equations of $\mathbb{R}P^n$ rotate we note that

Note 48 complexifies as follows:- If coefficients and the unknown $\mathtt{x}$ take values in $\mathbb{C}$ and we use its addition and multiplication, then $\mathbb{C}P^\infty$ is the space of all equations, $\mathbb{C}P^n$ all of degree $\leq n$, and $\mathbb{C}^n$ all of degree $= n$. This gives us a partition $\mathbb{C}P^n = \mathbb{C}^n \cup \ldots \cup \mathbb{C}^j \cup \ldots \cup \mathbb{C}^0$ of complex projective n-space into $n+1$ cells, one of each even dimension $0 \leq 2j \leq 2n$, which shows that its nonzero integral cohomology groups are $H^{2j}(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}$. Replacing $\mathtt{x}^j$ by $\mathtt{x}^j\mathtt{y}^{n-j}$ by $\mathtt{y}^{n-j}$ this oriented closed $2n$-manifold is equally the space of all homogenous equations in $\mathtt{x},\mathtt{y}$ of degree n, or else of all in $\mathtt{y}$ of degree $\leq n$, with the partition $\mathbb{C}P^n = {}^*\mathbb{C}^0 \cup \ldots \cup {}^*\mathbb{C}^{n-j} \cup \ldots \cup {}^*\mathbb{C}^n$, where ${}^*\mathbb{C}^{n-j}$ is all equations in $\mathtt{y}$ of degree $n-j$, the poincaré dual of the previous cell subdivision.

Over $\mathbb{C}$ any equation factorizes into degree one equations $y\mathtt{x} - x\mathtt{y} = 0$, i.e., any $\odot \in \mathbb{C}P^n$ has n extended complex roots $[x, y] \in \mathbb{C}P^1 = S^2$ counted with multiplicity, i.e., the closure of the projective complex n-swallowtail is all of $\mathbb{C}P^n$. All isomorphisms $\begin{bmatrix} v & -t \\ -s & u \end{bmatrix} \in GL(2, \mathbb{C})$ of $\mathbb{C}^2$ preserve its orientation, those which preserve each complex line through the origin are diagonal with entries same, so the isomorphisms of the extended complex numbers S^2 are in bijective correspondence with pairs $\pm A \in SL(2, \mathbb{C})$. The nth symmetric power of this action of $SL(2, \mathbb{C})$ on S^2 coincides with that by substitutions $\mathtt{x} \mapsto u\mathtt{x} + t\mathtt{y}, \mathtt{y} \mapsto s\mathtt{x} + v\mathtt{y}$ on $\mathbb{C}P^n$ as the space of all degree n homogenous equations in $\mathtt{x}$ and $\mathtt{y}$.

The isomorphisms of the extended complex numbers having real matrices $SL(2, \mathbb{R}) \subset SL(2, \mathbb{C})$ preserve the circle of extended real numbers $S^1 \subset S^2$, the two components of $S^2 \setminus S^1$, and act symmetrically on conjugate pairs $[x, 1], [\overline{x}, 1]$. When an equation $\odot \in \mathbb{R}P^n \subset \mathbb{C}P^n$ having such a pair of roots rotates to $\theta(\odot)$, these roots rotate to $[x \cos\theta - \sin\theta, x \sin\theta + \cos\theta] = [\frac{x\cos\theta - \sin\theta}{x \sin\theta + \cos\theta}, 1]$ and its conjugate. If $0 < \theta < \pi$ then $\frac{x\cos\theta - \sin\theta}{x \sin\theta + \cos\theta} = x \iff x = \pm\, i$, so $S^2 \setminus S^1$ gets partitioned into conjugate pairs of simple closed orbits around the roots $[\pm i, 1]$ of the unique equation $\mathtt{x}^2 + \mathtt{y}^2 = 0$ of $\mathbb{R}P^2$ which stays put.

Therefore : this action of the rotations of the extended real numbers S^1 on $\mathbb{R}P^n$ is free if n is odd, but for n even this flow has as a unique stagnation point, viz., the equation $(\mathtt{x}^2 + \mathtt{y}^2)^{n/2} = 0$. Further, this sole singularity is of degree one–confirming that the euler characteristic $e(\mathbb{R}P^n)$ is zero for n odd and one for n even–because it has invariant links. Indeed, on an invariant link, the orbits of this baby flow are the fibres of a hopf map $S^{n-1} \to \mathbb{C}P^{n/2}$. But, away from this one (for case n even only) fixed point, these orbits form a *foliation by circles* of $\mathbb{R}P^n$ which is far from any fibration, e.g., the order of the holonomy group of the cuspidal orbit is the l.c.m. of $0 < n - 2k \leq n$. However, we'll consider this natural foliation – which reminds us of similar, but far more intricate foliations of Epstein, Sullivan, Margulis, etc. – further elsewhere.

Unlike $SO(2, \mathbb{R}) \subset SL(2, \mathbb{R})$ which acts transitively on the extended real numbers $S^1 \subset S^2$, the complexified matrices of $SO(2, \mathbb{C}) \subset SL(2, \mathbb{C})$ do not act transitively on the extended complex numbers S^2. For, the same calculation shows that $\pm i$ remain fixed even if we analytically extend $\cos\theta$ and $\sin\theta$ to all $\theta \in \mathbb{C}$. But now the parallel circular $SO(2, \mathbb{R})$ orbits filling the complement of $\pm i$ merge into a single orbit of this bigger two dimensional noncompact group. That is, all points of S^2 other than $[\pm i, 1]$ are in the $SO(2, \mathbb{C})$-orbit of the point $\infty = [1, 0]$, that is, the meromorphic function $x = -\cot\theta$ wraps the complex plane $\mathbb{C}$ with period π on $S^2 \setminus \pm i = (\mathbb{C} \cup \infty) \setminus \pm i$:- Because $-\cot(it) = i\frac{e^{-t} + e^t}{e^{-t} - e^t}$, it follows that the image of the imaginary axis $\theta = it, -\infty < t < \infty$, under $-\cot$ is an open arc of S^2 which goes from $+i$ via ∞ to $-i$, hence it intersects all the circular $SO(2, \mathbb{R})$-orbits. $\square$

A similar argument of Picard using $SL(2, \mathbb{R})$ shows more generally that, no nonconstant meromorphic function can omit more than two values, and there are like results of Nevanlinna, Weyl, Ahlfors, etc., for holomorphic functions into the other riemann surfaces $M^2 \neq S^2$ or $\mathbb{C}P^n, n > 1$. In fact in some of the previous notes too, many others also deserved to be named, that I've alas not named. To some extent because I did not want all these names to scare you, and maybe myself as well, off a straightforward path. But verily : we are now at the crossroads of *many* beautiful theories ! Led by that little girl skipping along, we have, starting from a forgotten composition of long ago, reached this Charing Cross of today so easily only because we took a very natural way : just sprinkled some addition and multiplication into the cartesian creation and evolution of PG&R ! During a ramble there are many thoughts that come to mind, and then tend to slip away; the next note 50, which I plan to keep open-ended, will attempt to list some of these. And hopefully, if the Almighty so wishes, this ramble will continue on for some more time.

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