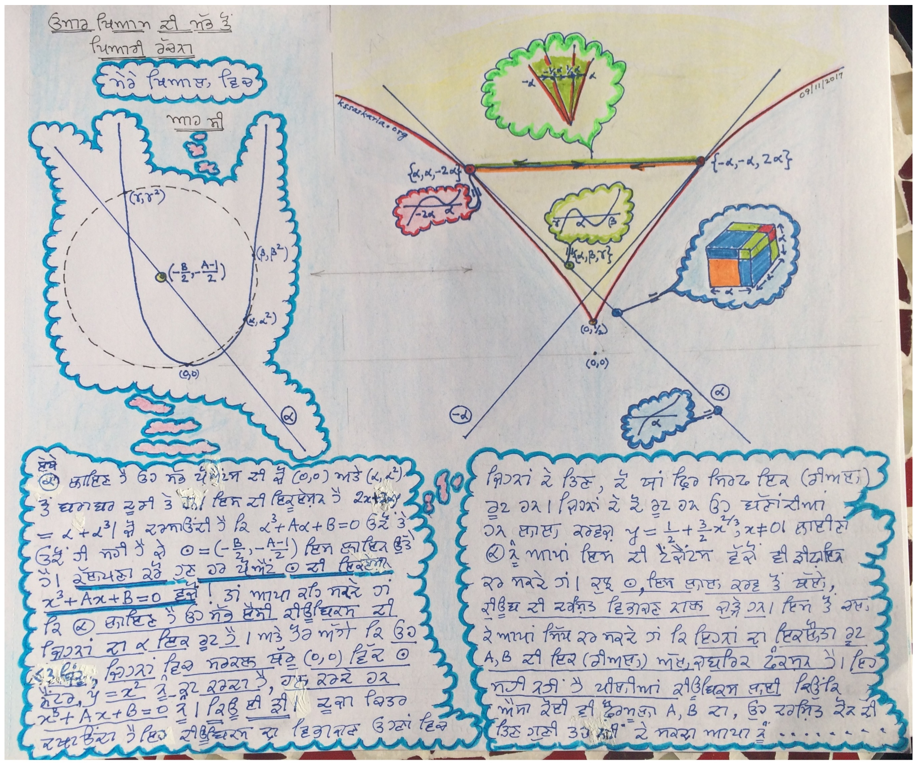
[Omar Khayyam's
prettiest composition
in my opinion
was this]



[Here $\textcircled{\alpha}$ is the line of points equidistant from $(0,0)$ and (α, α^2). Its equation is $2x + 2\alpha y = \alpha + \alpha^3$. Which shows $\alpha^3 + A\alpha + B = 0$ holds iff $\odot = (-\frac{B}{2}, -\frac{A-1}{2})$ is on this line. Imagine now each point $\odot$ as the equation $\mathbf{x}^3 + A\mathbf{x} + B = 0$.[1] Then we can say *the line* $\textcircled{\alpha}$ *consists of all such cubics having* α *as one root.* And furthermore that, *the* ≤ 3 *cuts on* $y = x^2$ *made by the circle through* $(0,0)$ *with* $\odot$ *as center solve* $\mathbf{x}^3 + A\mathbf{x} + B = 0$. q.e.d. The second picture shows the partition of all these cubics into

those having three, two or only one (real) root. Those with two roots form the red curves $y = \frac{1}{2} + \frac{3}{2}x^{\frac{2}{3}}, x \neq 0$. We can define the $\textcircled{\alpha}$'s also as its tangent lines. *Of the cubics* $\odot$ *below the red curves some arise from the shown decomposition of a cube.* Starting from this we can prove *their sole root is a (real) al jabric function of* A, B. *This is not true for the yellow cubics,* because such a function of A, B cannot triple-fold a cone in the manner shown ...]

[1] Unknown $\mathbf{x}$ is to be distinguished from the first coordinate of the points (x, y) of the plane.

Notes

1. Yes, I have read all his quatrains (as translated by Fitzgerald) and indeed some of these compositions are exquisite, but to my mind, *Khayyam's circle method* – see first picture – is far more beautiful than all of them.

2. How and why *it solves any cubic* $\mathtt{x}^3 + A\mathtt{x} + B = 0$ is proved in just a few lines penned below this picture. Here $B \neq 0$, so no root is zero, and the ≤ 3 cuts on $y = x^2$ that count are those other than $(0, 0)$.

3. *The 'quadratics' $B = 0$ form the y-axis.* If $\odot$ is on this line the circle is tangent to $y = x^2$ at $(0, 0)$ and it now counts as a cut. If also $A = 0$ then its radius $\frac{1}{2}$ equals the radius of curvature of $y = x^2$ at $(0, 0)$.

4. *Each line $2x + 2\alpha y = \alpha + \alpha^3, \alpha \neq 0$ of cubics $\odot$ has two with two roots,* $(\mathtt{x} - \alpha)(\mathtt{x} + \frac{1}{2}\alpha)^2 = 0$ and $(\mathtt{x} - \alpha)^2(\mathtt{x} + 2\alpha) = 0$, i.e., $\mathtt{x}^3 - \frac{3}{4}\alpha^2\mathtt{x} - \frac{1}{4}\alpha^3 = 0$ and $\mathtt{x}^3 - 3\alpha^2\mathtt{x} + 2\alpha^3 = 0$, i.e., the points $(\frac{1}{8}\alpha^3, \frac{1}{2} + \frac{3}{8}\alpha^2)$ and $(-\alpha^3, \frac{1}{2} + \frac{3}{2}\alpha^2)$ on the red curves $y = \frac{1}{2} + \frac{3}{2}x^{\frac{2}{3}}, x \gtrless 0$. *Furthermore, it is tangent at the second point,* because the derivative $dy/dx = x^{-1/3}$ here is equal to its slope $-1/\alpha$.

5. Omar's method tells us that, *the function whose values at each cubic $\odot$ are all its real roots, its graph $G \subset \mathbb{R}^3$ is the disjoint union of lines α^** parallel to @ at height α. So this surface cuts $\mathbb{R}^2$ in the y-axis, is homeomorphic to the plane, and has equation $2x + 2zy = z + z^3$.

6. The closed *half lines* of @ in the complement of that yellow *'peacockfeather'* are disjoint. The union of the half lines of α^* above them gives a homeomorphic closed subset of G, whose boundary comprises of the point $(0, \frac{1}{2}, 0)$ and one each of the two curves of G above each red curve. If we now delete the other two curves from the complementary open set of G we are left with, *three copies of the peacockfeather all tied to it by $(x, y, z) \mapsto (x, y)$.*

7. There is no line @ parallel to the x-axis, and one and only one in any other direction, so, *the curves $y = k$ of G cut each line α^* once and only once.* For $k \leq \frac{1}{2}$ these curves are graphs of single-valued strictly increasing functions z of x, but *for $k > \frac{1}{2}$, a shape S is born in their middle which then just keeps on growing with k.*

8. So, *all cubics $\odot$ cannot be solved by (real) 'al jabr'*, that is, there is no (finite) formula of A and B, equivalently $z(x, y)$ of x and y, built using only the first five operations of $\mathbb{R}$ – addition, subtraction, multiplication, division, surds – which has the graph G. Because, this S shape has no homeomorphism of order two which preserves its projection on the segment.

9. *'Completing a square'* tells us what G looks like above any @. The equations on this line are $\mathtt{x}^3 + A\mathtt{x} + B = (\mathtt{x} - \alpha)(\mathtt{x}^2 + \alpha\mathtt{x} + A + \alpha^2) = 0$. The roots of the quadratic factors make a parabola whose vertex, if $\alpha \neq 0$, is situated above the first red curve at height $-\alpha/2$, and whose two points above the second red curve are at heights α and -2α. The union of this parabola and the line α^* gives us all the points of G above @.

10. *'Completing a cube'* translates any equation to eliminate its second term, and amusingly our remaining hope – *all cubics $\odot$ in the lower half of the picture*

can be solved by al jabr – is also realized by this method :-

11. *All binomial decompositions of α^3 form that half line* ◎. $\alpha^3 = (u+v)^3$ minus three boxes $3u^2v$ and three $3uv^2$, so $3uv\alpha$, leaves cubes u^3+v^3, so α is a root of the $\odot$ with $A = -3uv$ and $B = -u^3 - v^3$. Further, since $u^3v^3 = -A^3/27$, *α is the sum of the cube roots of the roots of* $\mathtt{x}^2 + B\mathtt{x} - A^3/27 = 0$. This map $\odot(u,v) = (\frac{u^3+v^3}{2}, \frac{3uv+1}{2})$ *double folds the u-v plane symmetrically along its diagonal* $u = v$ to give the second picture minus the peacockfeather.

My goal was to update by notes this year only this story starting with Omar, but now—if God so wills—remaining notes next year. 31 December, 2017.