The fundamental theorem of algebra

This installment of parallel notes is around an aside [1] that arose in the last installment of *PG&R*. All books of algebra are replete with homomorphisms but homeomorphisms and manifolds are taboo! Yet, *the fundamental theorem of algebra is all about a homeomorphism and a property special to manifolds* [2] *of dimension two : their symmetric powers are also manifolds.* Once this, at first sight surprising, property is in our hands, the theorem follows easily, as we'll show in the very next paragraph. Then we'll directly work out the links in the symmetric powers of all manifolds. The method we use is tied to many diverse topics, and gives rise to many natural questions ...

(ਦ) *That the* `symmetric powers of two dimensional manifolds are manifolds` *implies, and is implied by, the fundamental theorem of algebra* :- We note that the closed $2n$-manifold $\mathbb{C}P^n$ *is the space of all degree n equations $a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n = 0$ over $\mathbb{C}$,* so $S^2 = \mathbb{C} \cup \infty = \mathbb{C}P^1$ is all degree one equations $lx + my = 0$. Multiplying n such equations gives us an injective map $\Psi : S^2 \star \cdots \star S^2 \to \mathbb{C}P^n$ from its symmetric power; so, if we know this power is a closed $2n$-manifold, `Ψ is a homeomorphism.` The restriction of Ψ to all unordered n-tuples of the subspace $\mathbb{C}$ of S^2 defined by $l \neq 0$ is `the fundamental homeomorphism` $\mathbb{C}\star\cdots\star\mathbb{C} \to \mathbb{C}\times\cdots\times\mathbb{C}$ `of algebra` with all its ordered n-tuples or $\mathbb{C}^n$ the subspace of $\mathbb{C}P^n$ defined by $a_0 \neq 0$. Conversely, any even dimensional manifold is locally like $\mathbb{C}^n$, so this homeomorphism implies the symmetric powers of 2-manifolds are manifolds. □

Likewise the open $2n$-manifold $\mathbb{C}^n$ *is the space of all degree n equations* $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, so $\mathbb{C}$ is all degree one equations $lx + m = 0$, and multiplying n of them gives the injective map $\Psi : \mathbb{C} * \cdots * \mathbb{C} \to \mathbb{C} \times \cdots \times \mathbb{C}$. To show its surjectivity we had `homogenized` it above, which is what enabled us to make use of the fact that, *between closed manifolds of the same dimension there are no proper inclusion relations*; besides, it showed us that *the nth symmetric power of S^2 is homeomorphic to $\mathbb{C}P^n$.*

(ਧ) `A symmetric power of` $\mathbb{R}^m$ `is a manifold iff` $m = 2$:- All sequences $(x_1, .., x_n), n \geq 2$, from $\mathbb{R}^m$ form $\mathbb{R}^m \times \cdots \times \mathbb{R}^m$; transpositions $x_i \leftrightarrow x_j$ define a binary relation on it, the generated equivalence classes $\pi(x_1, .., x_n)$ together with their open unions form the quotient space $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$.

If the $x_i \in \mathbb{R}^m$ are distinct there exist n closed m-balls B_i around them that are disjoint, so the restriction $\pi : B_1 \times \cdots \times B_n \to \pi(B_1 \times \cdots \times B_n)$ is injective, therefore a homeomorphism, which shows such points $\pi(x_1, .., x_n)$ of $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$ form an open mn-dimensional manifold.

If, but for $x_1 = x_2 = 0$, the x_i are distinct, choose $n-1$ disjoint closed balls B_i around them with $B_1 = B_2 = B$, then the restriction $\pi : B \times B \times B_3 \cdots \times B_n \to \pi(B \times B \times B_3 \cdots \times B_n)$ is such that $\pi(y_1, y_2, y_3, \ldots, y_n) = \pi(z_1, z_2, z_3, \ldots, z_n)$ iff $\{y_1, y_2\} = \{z_1, z_2\}$ and $y_i = z_i \forall i \geq 3$. Using $(y, z) \mapsto ((\frac{y+z}{2}, \frac{y+z}{2}), (\frac{y-z}{2}, \frac{z-y}{2}))$,

[1] See *Aliens and invisibility,* fn.20, also ਬੀਜ ਗਣਿਤ ਦਾ ਮੂਲ ਮਸਲਾ.

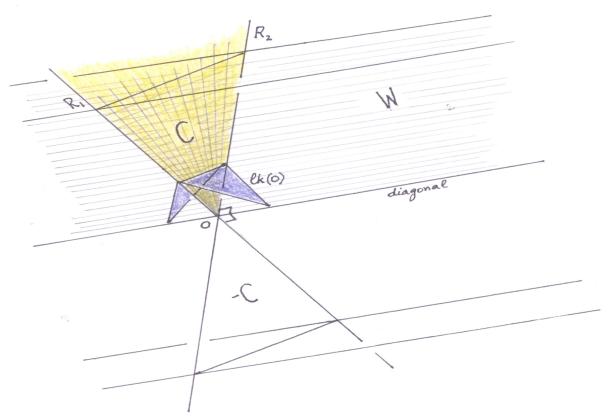
[2] Connected sans boundary, `closed` means compact, `open` if not.

a homeomorphism of $B \times B$ with $\Delta \times \Delta^\perp$, where $\Delta = \{(y,z) \in B \times B : y = z\}$ and $\Delta^\perp = \{(y,z) \in B \times B : y + z = 0\}$, we see *the link of* $\pi(0,0,x_3,..,x_n)$ *in* $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$ *is* $S^{nm-m-1} \cdot \mathbb{R}P^{m-1}$*,* viz., the join of $\partial(\Delta \times B_3 \times \cdots \times B_n)$ and $\partial\Delta^\perp$ mod its antipodal action : since $\mathbb{R}P^{m-1}$ `is homeomorphic to` S^{m-1} `iff` $m = 2$`,` it follows that for $m \neq 2$ these are singular points. □

(ਠ) `The link of` $\pi(x_1, x_2, x_3, .., x_n)$ `in` $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$—we'll push the above direct method to verify [3] this—`is the join` S^{nm-1} `of` n `spheres` S^{m-1} `if the` x_i `are distinct, otherwise replace` $\Sigma_x(n_x - 1)$ `of these` S^{m-1} `by` $\mathbb{R}P^{m-1}$ `where` n_x `denotes the number of` $x_i = x$`.` This follows by an easy induction on n – for we can separate these distinct values x of the x_i by small neighbourhoods as above – provided we can `prove :` *all points of* $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$ *below the* `diagonal` $x_1 = \cdots = x_n$ *of* $\mathbb{R}^m \times \cdots \times \mathbb{R}^m$ *have as their links the join of one sphere* S^{m-1} *and* $n - 1$ *projective spaces* $\mathbb{R}P^{m-1}$*.*

(ਪ) *Case* $m = 1$:- Since each equivalence class $\pi(x_1, \ldots, x_n)$ has a unique non-decreasing sequence, π *maps* W*, the* `chamber` $x_1 \leq \ldots \leq x_n$ *of* $\mathbb{R} \times \cdots \times \mathbb{R}$*, homeomorphically onto* $\mathbb{R} \star \cdots \star \mathbb{R}$*.* All inequalities strict give its interior, just one equality $x_i = x_{i+1}$ its $n - 1$ `walls`, more its lower dimensional faces, and $x_1 = \cdots = x_n$ the diagonal in which they all intersect. Any diagonal point O is at distance $\epsilon > 0$ from two other points—an S^0—on it, but from only one point—an $\mathbb{R}P^0$—on the normal R_i through it in the two-dimensional face $x_1 = \cdots = x_i \leq x_{i+1} = \cdots = x_n$. Since W *is the convex hull of these* $n - 1$ *faces,* the link of O is the join of an S^0 and $n - 1$ points $\mathbb{R}P^0$. □

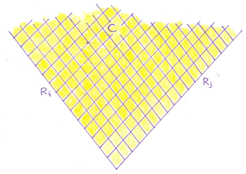
Taking O as origin $(0, \ldots, 0)$ these $n - 1$ rays R_i have as convex hull the cone C in which W intersects $x_1 + \cdots + x_n = 0$. The union $\tilde{W} = W \cup -W$ of lines through the full cone $\tilde{C} = C \cup -C$ parallel to the diagonal consists of all monotonic sequences, and identifying antipodal points of each normal-to-the-diagonal section of this gives back $W \cong \mathbb{R} \star \cdots \star \mathbb{R}$.



The link of O in each full line $\tilde{R}_i$ is S^0, in $\tilde{C}$ a subset of their join, in C join of as many $\mathbb{R}P^0$s, more generally, *the link of any point* P *has as many* $\mathbb{R}P^0$*s*

[3]This gives *links of the symmetric powers of any* m*-manifold*, and for $m = 2$ this direct verification of no singularities *proves* (ਟ) fundamental theorem of algebra.

in it as the number of `parallels` $\tilde{R}_{i,P}$ *through it which end at P,* and we'll describe now how this simple picture extends to all $m \geq 1$:-



(ढ) We'll write any $z \in \mathbb{R}^m$ as $(x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$, and let $W^{(m)}$ be all sequences $(z_1, \ldots z_n) \in \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ with *first coordinates non-decreasing,* $x_1 \leq \ldots \leq x_n$, and $C^{(m)}$ its intersection with the codimension m subspace $z_1 + \cdots + z_n = 0$. Each equivalence class $\pi(z_1, \ldots, z_n)$ has one member in $W^{(m)}$, and only one if in interior, but for $m > 1$ can have more if on boundary, so identifications need to be made here to get $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$.

For each i the extra constraints $z_1 = \cdots = z_i$ and $z_{i+1} = \cdots = z_n$ give an m-dimensional subspace $\tilde{R}_i^{(m)}$ of $z_1 + \cdots + z_n = 0$. These $n-1$ subspaces intersect in the origin, and have closed halves $R_i^{(m)}$, given by $x_i \leq x_{i+1}$, of which $C^{(m)}$ is the convex hull. The identifications above each $P \in C$ are determined by the $n-1$ parallel flats $\tilde{R}_{i,P}^{(m)} = P + \tilde{R}_i^{(m)}$: *we need to identify pairs of points antipodal with respect to P in each $\tilde{R}_{i,P}^{(m)}$ iff $\tilde{R}_{i,P}$ ends at P,* i.e., iff the parallel $\tilde{R}_{i,P}$ does not remain in C on both sides of P. Indeed, if P is in a wall $x_1 < \cdots < x_i = x_{i+1} < \cdots < x_n$ only $z_i \leftrightarrow z_{i+1}$ matters, it keeps fixed all points above P with $y_i = y_{i+1}$, and is antipodal on the complementary flat with $y_i + y_{i+1} = 0$ and all other $y_j = 0$, i.e., points above P in $\tilde{R}_{i,P}^{(m)}$. This gives us all identifications above a P in a wall. Otherwise, P is incident to some walls, and we need to identify above it all pairs of points which occur as limits of identified points above a $Q \to P$ in any of these incident walls.

Let $C_P^{(m)}$ be all points of $C^{(m)}$ lying in the flat with center P spanned by all $\tilde{R}_{i,P}^{(m)}$ such that $\tilde{R}_{i,P}$ ends at P, and $\tilde{C}_P^{(m)}$ its symmetrization with respect to P. The link – all points at distance $\epsilon > 0$ – of P in each of these $\tilde{R}_{i,P}^{(m)}$s is an S^{m-1}, but only a subset of their join is in $\tilde{C}_P^{(m)}$. Identifying antipodally with respect to P gives as many $\mathbb{R}P^{m-1}$s but some pairs of points in distinct $\mathbb{R}P^{m-1}$s are joined by not one but two segments. That is because so far we have only made identifications above P, identifications above all points in a neighbourhod of P identify such segments : the link of P in the identification space $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$ is the join of a sphere and all these $\mathbb{R}P^{m-1}$s, so none if P is an interior point, and maximum $n-1$ if $P = O$ when $C_O^{(m)} = C^{(m)}$. $\square$

Briefly we married the idea, first coordinates non-decreasing, which reduces us to the $n-1$ transpositions, $z_1 \leftrightarrow z_2, \ldots, z_{n-1} \leftrightarrow z_n$, with (प), which gives all identifications on the corresponding walls. *Their continuous extensions suffice:* e.g., above O any $(y_1, \ldots, y_i, y_{i+1}, \ldots y_n) = (y_1, \ldots, \frac{y_i + y_{i+1}}{2}, \frac{y_i + y_{i+1}}{2} \ldots, y_n) + (0, \ldots, \frac{y_i - y_{i+1}}{2}, \frac{y_{i+1} - y_i}{2}, \ldots, 0)$ is the limit of the corresponding point above a Q

in the wall $x_i = x_{i+1}$, but this we identified to $(y_1, \ldots, \frac{y_i+y_{i+1}}{2}, \frac{y_i+y_{i+1}}{2}, \ldots, y_n) + (0, \ldots, \frac{-y_i+y_{i+1}}{2}, \frac{-y_{i+1}+y_i}{2}, \ldots, 0) = (y_1, \ldots, y_{i+1}, y_i, \ldots y_n)$. So above O too the points $(y_1, \ldots, y_i, y_{i+1}, \ldots y_n)$ and $(y_1, \ldots, y_{i+1}, y_i, \ldots y_n)$ get identified, and since any transposition is a composition of these $n-1$, we have identified above O all the $n!$ permutations of this sequence.

We note that, $\mathbb{R}^m \star \cdots \star \mathbb{R}^m$ *is homeomorphic to the diagonal times the union* $C(X)$ *of coincident rays through the link* X *of* O *in* $C^{(m)}$. That this X splits as a join of $n-1$ projective spaces of dimension $m-1$ can be laid to the remark above regarding continuous extensions. Likewise if a point lies in a codimension t face of C its link has t projective spaces. Then $t = \sum_x (n_x - 1)$ of (ਠ), which also gives the number of projective spaces in the links of points of $C^{(m)}$ not in C that have some z_i same. This number is less than $n-1$ because no such point has all z_i same. For $m = 2$, and only for $m = 2$, these singularities are illusory, they disappear because $\mathbb{R}P^1$ is homeomorphic to S^1.

(ਬ। A conversation with Keerti was crucial ... it took us back to the 1990s and a striking result of topology—the nth symmetric power of S^2 is $\mathbb{C}P^n$—that apparently was in print only as its second-last exercise in Shafarevich, *Basic Algebraic Geometry* (1977) ... but amazingly, all that we had needed to do it was the fundamental theorem of algebra. It did not take long to recollect this argument (ਟ) but then I thought, why not also work out the links directly, and, since I so wanted that peculiar 'fun' we mathematicians get by tormenting ourselves with beautiful riddles, I made only a cursory browsing of the usual suspects before I went full-tilt. The browsing had given nothing, save that idea of defining a chamber using a total order, from the beautiful book of Adams, *Lectures on Lie Groups* (1969). My hunch is that, if and when I do make a detailed search of the literature, nothing given above, except the possible errors, will turn out to be really new : at best I would have yet again made something 'well-known' [4] a little more well-known!

I'll conclude with a quick and random miscellany from the numerous thoughts that came to me during this quixotic adventure :-

੧) The fundamental homeomorphism of algebra $\mathbb{R}^2 \star \cdots \star \mathbb{R}^2 \to \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ is explicit only in this direction; as for its inverse, Brahmagupta's formula can be pushed only till $n \leq 4$, elliptic functions work for 5 and 6, and using automorphic functions Poincaré finally got it for all n. All this is 'well-known' – as it should be since this is work done more than a 100 years ago – but alas, so far I've not been able to understand this 'folklore' to my satisfaction.

੨) In view of the above, it may be useful to – say by using a one parameter family of homeomorphisms $\mathbb{R}P^1 \to S^1$ starting from doubling the angle – seek deformations of the fundamental homeomorphism with simpler inverses : one can construe this as deforming the complex multiplication on $\mathbb{R}^2$, so this idea has a Mochizuki-like ring to it.

੩) Figuring out the topology of the symmetric powers of $\mathbb{R}^m$ can be construed to be a generalization of the fundamental theorem of algebra which is only the case $m = 2$, and maybe for $m = 4$ it can be tied to the algebra of quaternions,

[4] In the sense of, *How I learnt some well-known folklore* (2010).

but it seems unrelated to Eilenberg and Niven, *The "fundamental theorem of algebra" for quaternions* (1944).

੪) That any point $\pi(x_1, \ldots, x_n)$ of the nth symmetric power $M^2 \star \cdots \star M^2$ of any 2-manifold is nonsingular follows from the FTA because we can find a cell $U \cong \mathbb{R}^2$ of M^2 containing its at most n points $\{x_1, \ldots, x_n\}$, so our point is in $U \star \cdots \star U \cong U \times \cdots \times U$. Thus, besides $\mathbb{C}P^n$, which arises from S^2, there is a host of $2n$-dimensional closed manifolds, and it should be 'fun' to work out their cohomology rings and other invariants.

੫) This will show how many of these closed 4-manifolds $M^2 \star M^2$ pass the 'numerological' criteria of Atiyah and Manton (2016) and so are associated in their sense to say the elements in Mendeleev's periodic table! In the mid-1970s I myself was enthralled with the idea of modelling microphysics by closed manifolds, but now I think it is more natural to understand first the cartesian creation and evolution of *all* closed manifolds, and only later narrow down to those that have been, or are likely to be, observed in an experiment.

੬) In our choice of the fundamental domain $C^{(m)}$ priority was given to first coordinates, more even-handed would be to use the tilted prism of sequences $(x_1, \ldots, x_n)$ with the sum of the m coordinates non-decreasing, $tr(x_1) \leq \ldots \leq tr(x_n)$. Also note that our arguments were affine, we worked only with parallels to the faces of the domain, and the distance invoked once can be cayley distance. Therefore this method should extend not only to all finite euclidean reflection groups, but also to their cocompact relativistic brethren preserving a light cone. Any M^2 is a quotient of a finite index subgroup of such a group (in many ways, and automorphic functions are those that are well behaved with respect to such groups) and this affine construction can likely be kept equivariant : this would give even a discrete geometrical, so essentially combinatorial, understanding of the above $2n$-manifolds $M^2 \star \cdots \star M^2$.

੭) For example a 10-vertex triangulation of $S^2 \star S^2$ starts a series triangulating the symmetric powers of S^2, but is there a special reflection group from which only $\mathbb{C}P^2_9$ is born? One hint that this may be so is that the deleted joins of $\mathbb{R}P^2_6$, $\mathbb{C}P^2_9$ and two (?) other simplicial complexes are spheres, but not of the highest possible dimension, for which case there is a classification.

੮) The argument (प) for $n = 2$ is basic for deleted products and joins, its extension here to $n > 2$, by marrying it with the idea of non-decreasing first coordinates, should have a bearing on tverberg theory, both continuous, which has reached a best possible result, and affine linear, where I think there should be an index formula counting the tverberg points.

੯) For $m = 2$ the mirror of a reflection is of codimension 2, using coordinates we can complexify or double the dimension of (this can also be done simplex by simplex) the cone of *PG&R,* but would then have a two dimensional time, with the technical advantage of losing the boundary.

੧੦) In one installment of *PG&R* there is a figure showing how cayley balls give rise to deleted joins, that also ties up with the construction here ...

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