The fundamental problem of algebra

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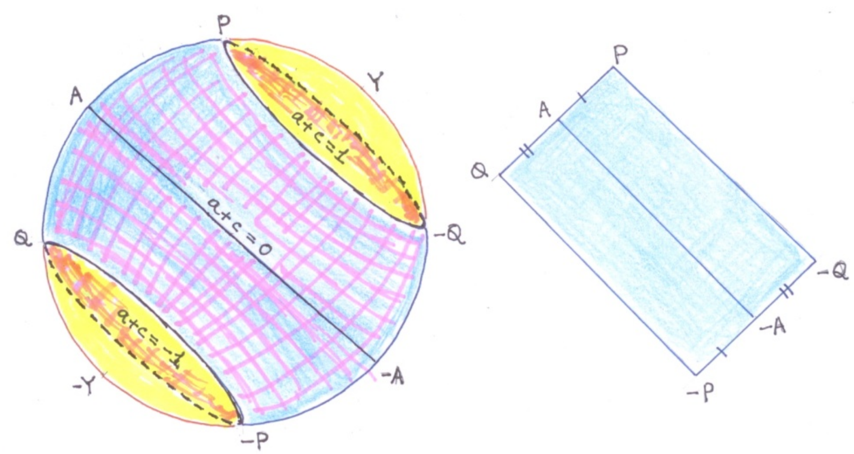
Theorem : *all equations which we solved in school using the quadratic formula, they form a Möbius strip!*

Proof. Complex numbers came after school, but we learnt when and how any second degree equation $ax^2 + bx + c = 0$ can be solved : multiply by 4a and 'complete squares' to rewrite as $(2ax + b)^2 - (b^2 - 4ac) = 0$ which showed it factorizes over the reals if and only if $b^2 - 4ac \geq 0$. We had also memorized the two solutions : the quadratic formula ! Homogenous second degree equations $ax^2 + bxy + cy^2 = 0$ in two unknowns had also appeared briefly, and the same method or formula was used to describe all their solutions under the same necessary and sufficient condition $b^2 - 4ac \geq 0$.

These homogenous second degree equations form the space RP^2 of all 3-tuples (a,b,c) not all zero with multiples (at,bt,ct) deemed same, i.e., the space of all lines through the origin of the space R^3 of all 3-tuples (a,b,c), i.e., the space of all pairs $\pm P$ of points in which they cut S^2 the surface defined by $a^2 + b^2/2 + c^2 = 1$. So equations with $b^2 - 4ac \geq 0$ form the subspace of RP^2 obtained from the shaded portion $-1 \leq a + c \leq +1$ of S^2 by identifying its pairs of antipodal points, i.e., the space obtained by glueing opposite edges of a strip of paper after a 180 degree twist. *q.e.d.*



The remaining equations form an open 2-cell -- the antipodal pairs of yellow points -- which attached to the boundary of this Möbius strip gives the RP^2. All degree one equations $ux + vy = 0$ form an RP^1 which is homeomorphic to S^1, so, *the quadratic formula gives us a homeomorphism from the Möbius strip to the symmetric square* $S^1 \bullet S^1$ *of a circle*, i.e., the space of all unordered pairs of its points. All quadratics $ax^2 + bx + c = 0$ identify with the subset $a \neq 0$ of RP^2, which is homeomorphic to R^2, and its subspace $b^2 - 4ac \geq 0$ is homeomorphic to a closed half plane, the quadratic formula gives us a homeomorphism of the same to the symmetric square $R \bullet R$ because all degree one equations $ux + v = 0$ identify with the subspace of RP^1 defined by $u \neq 0$ which is homeomorphic to R.

Many questions arise. Is there a similar dissection of the space RP^n of all homogenous degree n equations in two unknowns ? What is the symmetric nth power of a circle like ? Et cetera, but it is best to consider first such questions over the *complex numbers* C, for then not only the quadratic formula always works but there is also an amazing general theorem. Again, we'll think of C as all degree one equations $ux + v = 0$, likewise all degree n equations $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ is C^n, and, *multiplying* n *degree one equations to make a degree* n *equation gives us a continuous one-one map from the nth symmetric power* $C \bullet \ldots \bullet C \rightarrow C^n$: *the fundamental theorem of algebra says it is a homeomorphism!* The map is simple in terms of the complex multiplication we equip R^2 with to make C, and the conclusion so striking : *at first sight the nth symmetric power* $R^2 \bullet \ldots R^2$ *seems to have lots of singularities, the theorem assures us it has none!* This in fact is the heart of the matter, the theorem is equivalent to this absence of singularities : for this and much more see notes (ਟ – ਥ). This existence theorem comes with what I call *the fundamental problem of algebra* : *describe the inverse homeomorphism as explicitly as possible for all n.* This has engaged mathematicians for centuries, and in principle at least it seems that Poincaré had solved this problem more than a hundred years ago, but to the best of my knowledge, no one has written out a clear description of the inverse homeomorphism for all and sundry.

Briefly, Ruffini and Abel showed that quadratic-like formulae won't work for $n > 4$, but such formulae were known for $n \leq 4$, and could be written more simply using trigonometric, i.e., singly periodic functions. Abel and Jacobi then explicated the inverse for $n = 5$ using elliptic, i.e., doubly periodic meromorphic functions on the plane. The euclidean geometry of the plane does not allow more than two independent periods. The way out was to drop the infinitude of the plane, think of it as an open disk of radius $c < \infty$ -- see *Plain Geometry & Relativity* -- and admit all linear reflections which preserve the cone over it. This gives us many more reflections than in the classical or euclidean limit $c \to \infty$. This relativistic plane admits equally nice functions having any given number of independent periods, and the inverse homeomorphism was understood by Poincaré in terms of these automorphic functions.

Considering this complexity of its inverse, perhaps we should give up to some extent the direct homeomorphism, equivalently the complex multiplication on R^2, and try to deform it in such a way that the inverse becomes simpler ? This sounds like, but may or may not be, what Mochizuki has been doing recently …

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