The Joys of Forgetting

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December 31, 2012. There is a Poincaré recurrence at work in our garden of mathematical delights. If we keep on flitting in it fast enough, so forgetting even faster, we are doomed to enjoy its delights again and again! This was brought home to me by a recent request for some seminar notes from 1994. As I flipped through these old notes --- though I read a lot by *others*, somehow I find it hard to `go back in time' to re-read my *own* stuff, and my resolution for 2013 is indeed to at least *try* to do something about this ... --- I saw with a pang that there was much there that I had forgotten, but this sadness was soon made up by the beautiful memories that came flooding back.

This idiosyncratic seminar was driven by a compulsion to unmask the simplicity in the definitions of as many as possible of the diverse homologies that can be found in our garden. The functorial way then, and still now, for anyone who might be bitten by this bug, is to first read everything written by **Samuel Eilenberg**. There is an elegant simplicity in all his work—especially in papers of which he was the sole author—that has seldom been surpassed. Like so many of us, I too have striven for beauty, and it is for others to say by how much I have fallen short of the Eilenberg standard of excellence, but I dare say that, in one respect at least, I have gone neck-to-neck with this master. He wrote somewhere that he had never done five lines of any calculation without making a mistake, and this has remained a constant source of inspiration for me! The name of this great artist is indelibly linked to category theory, semi-simplicial complexes, obstruction theory, singular homology ... As a counter-point to the last, let's muse on a simple-minded

Non-singular homology. Singularities being taboo, we'll define this only for a smooth manifold. And i-chains shall be i-dimensional compact and oriented submanifolds M^i. So an i-chain has summands—its components—but we won't define sum to stay safely away from singularities. But we do have the oppositely oriented submanifold $-M^i$, and we can imagine for each i an empty i-dimensional submanifold 0 with both orientations. Which is handy, because ∂M^i, the (i-1)-dimensional manifold-boundary with the induced orientation, is not available in the really attractive case of a closed manifold, so setting $\partial M^i = 0$ for these, we have always $\partial\partial = 0$! Emboldened by this feat, we now replace any closed M^i which is the manifold-boundary of a submanifold by 0, and denote what we get by $H_i^{smooth}(\;\;; Z)$, even though this is merely a set with a distinguished element. However it should be mentioned here—though I forget the exact details at this exact point in time—that these ideas can be tweaked so that, for example, all the closed submanifolds of R^∞ get organized not only into an abelian group, but even a ring, and that the structure of this cobordism ring is known (for all the references see the Notes at the end, which I'll keep on refreshing for some time).

That this homology is different from singular homology is easy, for example, $H_1^{smooth}(R^2; Z)$ *is nonzero,* because the union M^1 of two like-oriented concentric circles in the plane is not the oriented boundary of the enclosed annulus with either orientation. On the other hand it is true that $H_1^{smooth}(R^3; Z)$ *is zero,* because any closed and oriented M^1 in space is the oriented boundary of an embedded surface F^2, which Seifert proved as follows.

In fact there is such a surface for each snapshot of the strings M^1 that is etched by centrally projecting them on a relatively much bigger round spherical general position screen which envelops them in space. As usual, we'll draw this *diagram*—a graph K with all vertices v of valence four—on a plane (so there is a point at infinity, also we'll assume that K is connected and with at least one vertex) and indicate by breaks near each vertex as to which of the two transverse roads is to *jump* over the other at this apparent crossing. The orientation of M^1 gives us a distinguished *state of* K, i.e., a disjoint partition of its edges into *circuits*, viz., the one obtained by starting out in the given direction and turning left or right`just before' each crossing as dictated by the direction on the transverse road. We keep only that portion of our spherical screen that is bounded by these circuits—this is a disjoint union of disks and is coloured blue in the picture below—and attach to it the twisted orange ribbons to obtain the desired surface F(K).

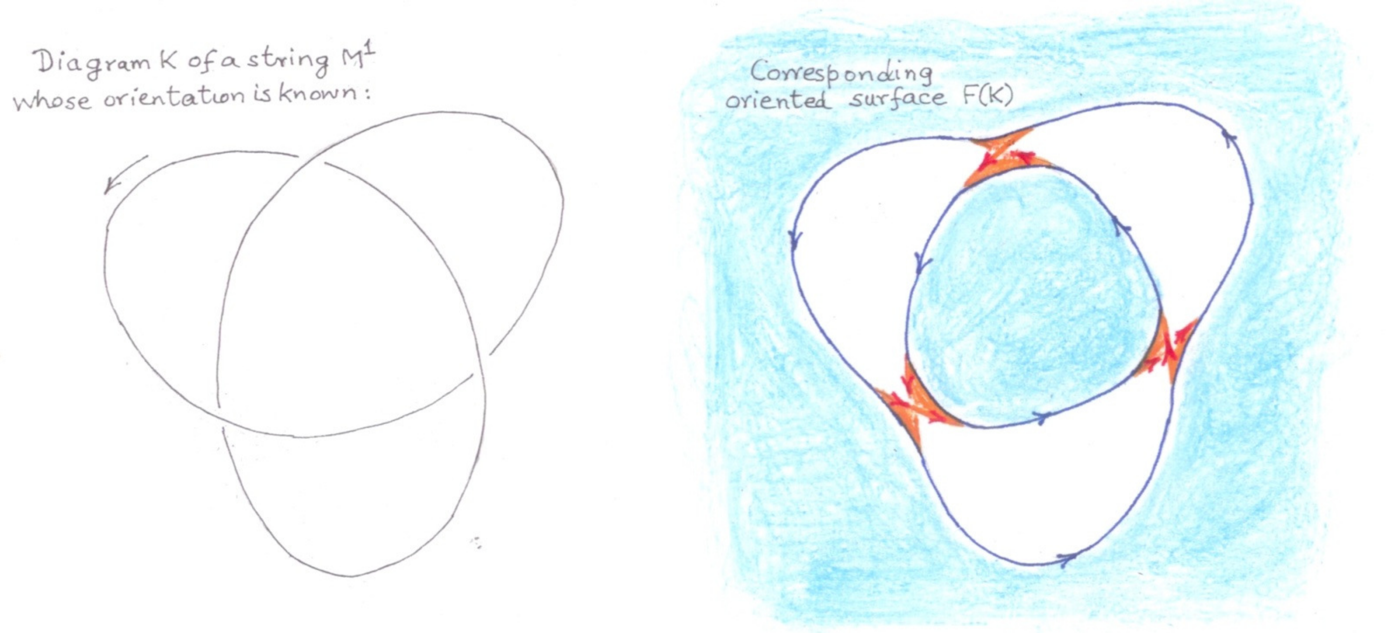


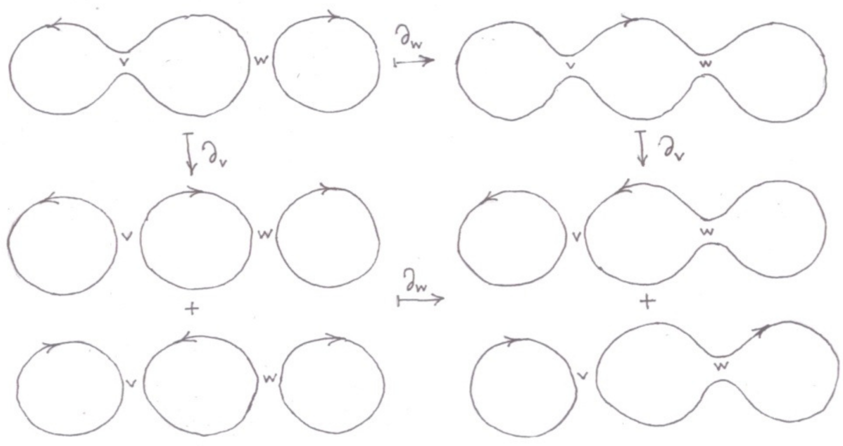
Figure 1

Thickening this oriented surface F(K) on either side we get two handlebodies with $\beta_1(F(K))$ = rank $H_1(F(K))$ holes, where we're now talking singular homology. We choose corresponding disjoint loops around these holes inside each handlebody, and denote by a_{ij} the number of times the ith loop of the first links with the jth loop of the second. Then det $[Xa_{ij} - a_{ji}]$ is—up to sign and multiplication by powers of X—an ambient isotopy invariant of oriented strings, called *Alexander's polynomial,* who had defined it, without using F(K), as a bigger determinant of size vert (K).

There are also interesting ambient isotopy invariants of strings that do *not* require any knowledge as to how they might be oriented, notably the following homology—which was not treated in that seminar for the very good reason that it was not even born then!—whose discoverer is perchance working in the same university in which Eilenberg had spent most of his mathematical life.

Khovanov's homology. The *ground states* of K shall be those in which, starting out in either direction we turn *left* `just before' each vertex v if we are about to jump, otherwise right. Besides, we'll consider the states that are obtained from these by *exciting*—that is by reversing the rule just enunciated—one or more vertices. The i-chains $C_i(K)$ are integral linear combinations of all states with precisely i excited vertices. If a vertex v is unexcited in, and occurs in two distinct but like, respectively unlike oriented circuits of a state s, then $\partial_v(s)$ shall be the state obtained by exciting v, and assigning to the resulting *fusion* of these two circuits the positive, respectively negative orientation (P = anticlockwise, N = clockwise in Figure 2). On the other hand, if this unexcited v occurs twice in the same positive, respectively negative circuit of the state s, then its excitation will *fission* this circuit into two parts, and we denote by $\partial_v(s)$ the sum of the two states obtained by assigning to these two parts unlike, respectively like, orientations

We totally order the vertices of K, and for any vertex v and state s, define $c_v(s) \in \{-1, 0, +1\}$ thus: if v is already excited in s then $c_v(s) = 0$, otherwise $c_v(s)$ is + 1 or -1 depending on whether the number of excited vertices in s less than v is even or odd. The linear operator ∂ on chains is defined by $\partial s = \Sigma_v\, c_v(s)\partial_v(s)$ for all states s, the summation being over all the vertices of K. One has $\partial\partial = 0$, that is, $\partial_v\partial_w(s) = \partial_w\partial_v(s)$ for any two vertices v and w which are unexcited in the state s. This is easy because we only need to follow what happens to the at most four circuits of s in which v and w can occur, indeed there is only one non-obvious case, for which this is depicted in the picture below.



$$\therefore \quad \begin{tikzcd} PN \arrow[r, mapsto] \arrow[d, mapsto] & N \arrow[d, mapsto, ";\text{etc.}"] \\ (PN+NP)N = P(NN)+N(PN) \arrow[r, mapsto] & PP+NN \end{tikzcd}$$

Figure 2

An i-chain has *filtration* ≥ j if the states s occuring in it obey $i + p(s) - n(s) \geq j$, where p(s) is the number of positive and n(s) the number of negative circuits of s. The defining fusion and fission rules of $\partial_v(s)$, viz.,

$$PP \to P \quad , \quad NN \to P \quad ; \quad PN \to N \; , \quad NP \to N$$

$$P \to PN + NP \qquad ; \qquad N \to PP + NN$$

show that then $\partial_v(s)$ also has filtration ≥ j, in fact, but for the highlighted items, which give some states in $\partial_v(s)$ having filtration ≥ j+4, the filtration of $\partial_v(s)$ would have been exactly the same as that of s.

This filtered chain complex is due to E S Lee (2005), and its spectral sequence is an ambient isotopy invariant of the strings M^1 from the first term onwards. Indeed it was this first term, i.e., the homology of the operator ∂_1 defined just like ∂, but we replace the highlighted items by zero, that had been introduced earlier by Khovanov (2000). This operator has bi-degree (1,0) if we put $C_i(K) = \Sigma_j\, C_{i,j}(K)$, where $C_{i,j}(K)$ is the linear combination of all states s with i excited vertices and $i + p(s) - n(s) = j$, and we'll denote by $H_{i,j}^{\text{Khovanov}}(K\,;Z)$ the corresponding summand of its homology.

So the integral polynomial in two variables $\Sigma_{i,j}\, X^i\, Y^j\, (\text{rank}\ H_{i,j}^{\text{Khovanov}}(K\,;Z))$ is an ambient isotopy invariant, in particular, the euler characteristic $\Sigma_{i,j}\,(-1)^i\, Y^j\,(\text{rank}\ H_{i,j}^{\text{Khovanov}}(K\,;Z)) = \Sigma_{i,j}\,(-1)^i\, Y^j\,(\text{rank}\ C_{i,j}(K))$ is one. The polynomial on the right of the last equation is a version of *Jones' polynomial,* an ambient isotopy invariant from 1984, which of course was the motivation for the definition of these strictly finer homological invariants.

In conclusion, about those seminar notes of 1994, which were so far circulating only as photocopies, I would like to thank You Qi (a co-author of Khovanov's who is also in the same department) for his kindness in making on his own, and sending back to me their pdf version, that you can now download conveniently from my website.

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Notes

1. As such the only thing worth knowing about $H_i^{smooth}(\ ; Z)$ is whether it is zero! If not, e.g. $H_1^{smooth}(R^2; Z)$, this set without any structure is huge, because even perturbations of a nonzero element are distinct. So there is an obvious need for 'tweaking' what we began to a bigger equivalence relation on submanifolds called a cobordism, but there are different cobordisms, so different results ... Whenever I have to bone up on cobordism theory—this I have done by now umpteen times!—I start the new cycle with Milnor's thin book, *Topology from the differentiable viewpoint* (1965), a gem if there is any. Our focus here being a different—as $H_1^{smooth}(R^3 ; Z) = 0$ underlines—organization of submanifolds, viz. that under ambient isotopy, we didn't go down this path; but note that even $H_1^{smooth}(R^3 ; Z) = 0$ had something of value to say about an invariant of ambient isotopy ... the two organizational principles are intertwined.

2. Alexander's polynomial was unveiled in *Topological invariants of knots and links,* Trans. Amer. Math. Soc. 30 (1928), 275-306, a beautiful paper which has, in my humble opinion, many ideas that are still unexplored; Seifert's finesse appeared six years later as *Ueber das Geschlecht von Knoten,* Math. Ann. 110 (1934), 571-592.

3. It is useful to not confine the use of the phrase, *state of* K, to those in which one turns either left or right at each vertex, for example, one has the important 2^t states—here t denotes the number of strings M^1—in which one continues travelling in the chosen direction on the same road past every vertex, which gives us all the possible orientations of M^1 itself. In E S Lee, *An endomorphism of the Khovanov invariant,* Advances in Math. 197 (2005) 554-586, it is shown that the final term of the spectral sequence is nonzero only for i = 0 and has rank 2^t. Though I have still to read this paper it seems to me that this can be proved by constructing a chain contraction to the trivial chain complex in which all nonzero chains have i = 0 and are linear combinations of the states just mentioned.

4. Those *n-circuits* of yore—see, for example, Lefschetz's *Introduction to Topology* (1949)—that have now become 'oriented n-dimensional simplicial pseudomanifolds' ring a bell in me that there is a possibility of generalizing all this to higher dimensions ... also there is that thing called oriented matroids of circuits ...

5. You will note that—in line with what Descartes taught us—I always speak of the strings as being oriented, only that there can be *uncertainty* in our knowledge about their orientation when seen under that spherical microscope. But then, we still have well-defined ground and excited states, so Khovanov's homology, as long as—in line with Kelvin's 19th century revival of Descartes' string theory—our observer can somehow make out the nature of those *infinitesimal* jumps in that snapshot K of these strings that has been etched on that screen by central projection.

6. My definition of *Khovanov homology* owes much to a 2011 paper with this title that is available on L H Kauffman's website and arXiv, however, I call Oleg Viro's enhanced states simply states, and in line with the above predilections, have decided that this 'enhancement' must be a choice of an orientation for each loop. More generally, I would like to mention here that I've found almost everything written by Kauffman to be helpful as I continue to flit around in this—almost totally new for me—part of our mathematical garden.

7. Talking of gardens, I was in fact working on another but related paper inspired by *Bina's garden*—which has trees and flowers and all ... —till uncertainty intervened: my old and faithful laptop decided to cease operations on the morning of December 14, and this year-end quickie has come into being instead!