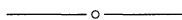


If  $a/b$  is very large, then  $x_0 = w/\sqrt{(a/b)^2 - 1}$  is close to zero, so the incorrect intuition that  $f$  is minimized at 0 becomes correct “in the limit.”

It is interesting to note that when  $a > b$  the *maximum* of  $f$  occurs either at 0 or  $l$ : at 0 if  $w \geq ((a^2 - b^2)/2ab)l$  and at  $l$  if  $w \leq ((a^2 - b^2)/2ab)l$ .



### Taylor’s Formula via Determinants

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For calculus students who know determinants one can, after doing Rolle’s theorem, proceed to the following

**Theorem.** Let  $f(x), f_1(x), \dots, f_{n+2}(x)$  be  $n + 1$  times continuously differentiable functions. Then

$$\begin{vmatrix} f(x) & f_1(x) & \dots & f_{n+2}(x) \\ f(0) & f_1(0) & \dots & f_{n+2}(0) \\ f'(0) & f_1'(0) & & f_{n+2}'(0) \\ & \dots & & \\ f^{(n)}(0) & f_1^{(n)}(0) & \dots & f_{n+2}^{(n)}(0) \\ f^{(n+1)}(h) & f_1^{(n+1)}(h) & \dots & f_{n+2}^{(n+1)}(h) \end{vmatrix} = 0 \quad (1)$$

for some  $h$  between 0 and  $x$ .

*Proof.* Consider  $x$  as constant and let  $D^{(i)}(h)$  denote the function of  $h$  obtained by replacing the last row of the determinant with  $f^{(i)}(h) f_1^{(i)}(h) \dots f_{n+2}^{(i)}(h)$ . Observe that for  $i = 0, 1, \dots, n$  the derivative of  $D^{(i)}(h)$  with respect to  $h$  is  $D^{(i+1)}(h)$  and the determinant in (1) is  $D^{(n+1)}(h)$ . Now  $D^{(0)}(0) = 0$  because the second and the last rows are the same; likewise,  $D^{(0)}(x) = 0$  because its first and last rows are the same. So, by Rolle’s theorem,  $D^{(1)}(h) = 0$  for some  $h$  between 0 and  $x$ . Also, the last row of  $D^{(1)}(0)$  is the same as its third. So, using Rolle’s theorem again,  $D^{(2)}(h) = 0$  for some  $h$  between 0 and  $x$ . Continuing, we see that  $D^{(n+1)}(h) = 0$  for a suitable  $h$  between 0 and  $x$ . *q.e.d.*

For example, (1) shows that for some  $h$  between 0 and  $x$ , we have

$$\begin{vmatrix} f(x) & 1 & \frac{x}{1!} & \frac{x^2}{2!} & \dots & \frac{x^n}{n!} & \frac{x^{n+1}}{(n+1)!} \\ f(0) & 1 & 0 & 0 & \dots & 0 & 0 \\ f'(0) & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \dots & & & \\ f^{(n)}(0) & 0 & 0 & 0 & \dots & 1 & 0 \\ f^{(n+1)}(h) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} = 0$$

which is Taylor’s formula because the determinant is

$$f(x) - f(0) - \frac{x}{1!} f'(0) - \frac{x^2}{2!} f''(0) - \dots - \frac{x^n}{n!} f^{(n)}(0) - \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(h).$$