Startling Logic

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April 10, 2013. Though $H_1^{SB}(S^2;Z/2) = 0$ has still to catch on in logical circles, there is a *"startling reason"* why the *round* sphere is already in : *there is no function* f *from unit* 3-*vectors to* {0,1} *which takes the value* 1 *exactly once on each orthonormal frame*! In other words, the restriction of f to any great circle should either (a) separate all pairs of points that are $\pi/2$ apart, or (b) be identically 0 on it. So for any $\epsilon > 0$ we'll have great circles—Figure 1—of types (a) and (b) with angular separation between them less than ϵ. Suppose they intersect in $\pm$**k**, and $\pm$**i** and $\pm$**v** are the midpoints of their semicircles—the great circle arc **iv** has length less than ϵ—and **j** is the point at distance $\pi/2$ from **i** on **iv** produced, so f($\pm$**i**) = 1 and f($\pm$**k**) = f($\pm$**v**) = f($\pm$**j**) = 0. The distance of a point A_1 on the great circle (a) between **i** and **k**, to a point C on the great circle through **k** and **j**, varies continuously from a value less than $\pi/2$ when C = **k**, to a value more than $\pi/2$ when C = −**k**. So, if ϵ is small, for some intermediate C = C_1, the great circle through A_1 and C_1 intersects the great circle (b) in B_1 such that $B_1C_1 = \pi/2$. Likewise, from the point A_2 between **i** and −**k** given by $A_1A_2 = \pi/2$, we have a great circle $A_2B_2C_2$ with $B_2C_2 = \pi/2$. But $f(C_1) = f(B_1) = 0$, so $f(A_1) = 0$, and likewise $f(A_2) = 0$, a contradiction.

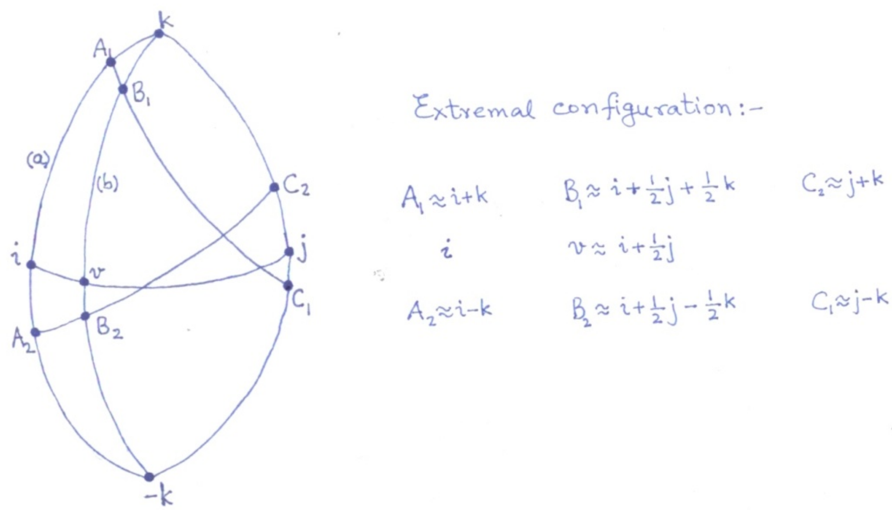

Figure 1

Further, the same ϵ works if A_1 is replaced by another point between **i** and **k** which is no closer to {**i**,**k**}. So, without changing **v**, we can take A_1 and A_2 to be in the directions **i**$\pm$**k**. Likewise, we can take C_1 and C_2 symmetric with respect to **j** in the directions **j**−x**k** and **j**+x**k**, where the positive number x depends on the angular separation **iv**. If this is $\tan^{-1}t$, i.e., if **v** is in the direction **i**+t**j**, t > 0, then B_1 is in the direction (**i**+**k**)+t(**j**−x**k**), and $B_1C_1 = \pi/2$ is equivalent to saying that the dot product of this vector with **j**−x**k** is zero, i.e., $tx^2 - x + t = 0$. This quadratic has real roots if and only if $t \leq ½$, so, *the above configuration exists if and only if the angular separation is at most* $\tan^{-1} ½$.

Given any **i** ≠ **v** on S^2 with **iv** $\leq \tan^{-1} ½$, there is a *finite* subset F of S^2 containing **i** and **v** such that no truth function f : F → {0,1}, *with* f(**i**) = 1 *and* f(**v**) = 0, takes the value 1 exactly once on each orthonormal frame in F. For, we only need to throw in **j**, **k**, A_1, A_2, B_1, B_2, C_1, C_2, and some cross products of orthogonal pairs of these. One also has *finite startling subsets* for which no extra condition on the truth functions is needed. Given any angles $0 < \theta_i < \pi$, we can draw (possibly with self-intersections) on S^2 a *connected* graph G with each edge a great circle arc having one of these lengths θ_i and with three of its vertices orthogonal to each other. Any f : vert (G) → {0,1} which takes the value 1 exactly once on each orthonormal frame will take different values on the vertices of at least one edge. So, if all $\theta_i \leq \tan^{-1} ½$, and we throw in, for each directed edge, some more points as above, we'll obtain a finite set of the required type. Also, if the graph G we use in this construction is mapped onto itself by an isometry of the round sphere, so is this finite set: therefore, any *finite group of isometries of* S^2 can be realized as the group of symmetries of such a finite set.

The bound $\tan^{-1} ½$ *is tied to the rani of these finite groups* : $\tan^{-1} 2$ is what any edge of a ***"Beesmukhi"*** subtends at its centre! For, an icosahedron can be made—Figure 2—from three congruent rectangles I, II and III, each inserted perpendicularly half-way through a parallel central slit of the next in cyclic order, but the distance of any vertex to the two nearest vertices of the next rectangle is equal to its breadth only if we use ***"Perfectly proportioned"*** rectangles, i.e., those whose length-to-breadth ratio x satisfies $x – x^{-1} = 1$, which is true if and only if $\tan(2\phi) = 2$.

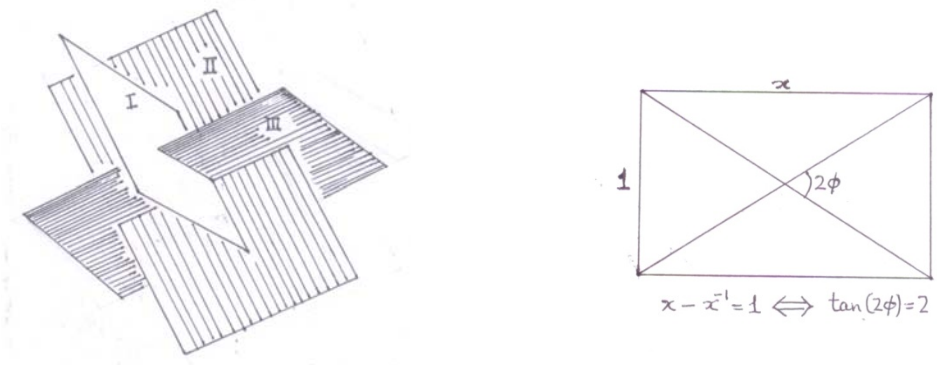

Figure 2

So, for each choice of orthogonal axes, this 'golden value' of x gives us an icosahedral *tiling* of S^2 into 20 regular 3-gons of side $\tan^{-1}2$, 5 at each of its 12 vertices in the directions (±x, ±1, 0), (0, ±x, ±1), (±1, 0, ±x). The distance 2ψ between face-centers of triangles sharing an edge, e.g. the angle between (x, 1+ 2x, 0) and (1+x, 1+x, 1+x), is $\sin^{-1} (2/3)$. Joining these adjacent face-centers gives us the dodecahedral *dual tiling* into 12 regular 5-gons of side 2ψ, 3 at each of its 20 vertices. These tilings share a common barycentric subdivision having 120 right angled triangles of sides $\{\phi, \pi/2 – \phi – \psi, \psi\}$. This set of right-triangles 'is' the finite group of all isometries of S^2 preserving the tiling : there is a unique symmetry mapping any chosen 'identity triangle' to any other triangle. These isometries preserve *the connected graph with equal edges of length* $\psi < \tan^{-1} ½$ *obtained by bisecting each side of the dodecahedral tiling*. Likewise, cutting each side of an octahedral tiling of S^2 into 4 equal parts gives us a connected graph with all edges $\pi/8$.

Metric homologies. More generally, we can consider the higher connectivities $\pi_i(K_t)$, homologies $H_i(K_t)$, etc., of the *simplicial complex* $K_t(S)$ with simplices all finite subsets, of a given metric space S, having points at distance ≤ t from each other. Playing around some more with the connectivity of graphs $G_t(F)$ on finite subsets F of the round 2-sphere with all edges of length ≤ t should convince you that these homological invariants of a metric are usually hard to compute. Yet an interesting definition of this kind may be flying in the wind ... It is known that the *homeomorphism type* of any compact space X – so, for example, the genus in the case when X is a closed surface – determines and is determined by the *isometry type* of the vector space C(X) of continuous real functions on X with the sup norm. *Is there a nice definition, using only the metric on the unit sphere* S(X) *of this banach space, which recovers the usual homology of* X? Though I don't know of such a definition ... there does come to my mind, as I write these words, that remarkable 1942 paper of Eilenberg ... and yes, the components $\pi_0(X)$ do translate to direct summands ... so maybe banach space techniques can shed some new light on the higher homotopy groups $\pi_i(X) = \pi_{i-1}(\Omega X)$ also?

Of course, we can look also at the cohomology of the complement of K_t or define our simplices by other metric conditions ... Indeed, in startling logic, the graph of choice H(F) on a finite subset F of S^2 is that whose edges are all pairs of points $\pi/2$ apart : *the more the number of edges in* H(F), *the closer the set* F *is likely to being startling*! As an example of this dictum consider, for any $0 < t < 1$, the set of 33 antipodal pairs of points on S^2 depicted in Figure 3 by means of the 66 points in these directions on a concentric cube of side 2 : the 12 mid-points of edges, and in each of the 6 faces, the 9 points having coordinates –t, 0 or +t with respect to its centre. If t = ½ this is the cubically symmetric subset of S^2 generated by the extremal configuration of Figure 1. However, it is at a *bigger* value of t, viz., at $t = 1/\sqrt{2}$, that H(F) has the most edges, for the dot products $(\mathbf{i} + t\mathbf{j} + t\mathbf{k}).(-\mathbf{i} + t\mathbf{j} + t\mathbf{k})$, etc., vanish only then. And, sure enough, it is only then that this finite set F is *almost startling*: it admits no truth function, *which is* 0 *on points of* F *orthogonal to those on which it is* 1, and which takes the value 1 exactly once on each orthonormal frame in F (the italicized condition is necessary, if we want to remove it some more points have to be added).

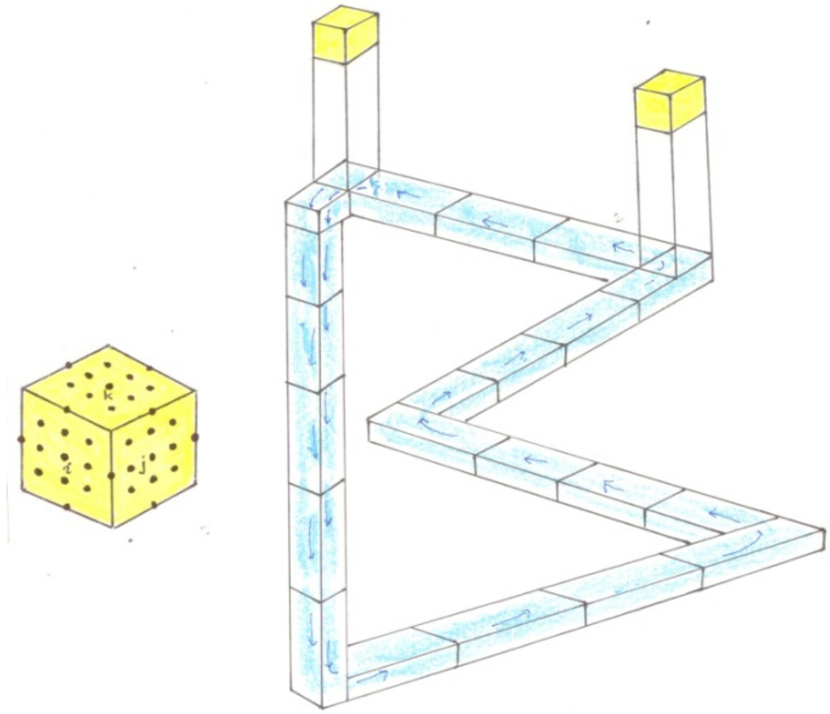

Figure 3

Alternatively, *these are the directions to the barycentres of the* 3 *cubes obtained by rotating this cube by* 45⁰ *about* **i**, **j** *and* **k**. For example, the rotation about **j** gives us the cube with vertices $\pm\sqrt{2}\mathbf{i} \pm \mathbf{j}$ and $\pm \mathbf{j} \pm \sqrt{2}\mathbf{k}$, and $\mathbf{i} + (1/\sqrt{2})\mathbf{j}$ is the direction to $\sqrt{2}\mathbf{i} + \mathbf{j}$, while $(1/\sqrt{2})\mathbf{i} + \mathbf{j} + (1/\sqrt{2})\mathbf{k}$ is the center of the edge joining $\sqrt{2}\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \sqrt{2}\mathbf{k}$, etc. It was pointed out by Penrose that an ornament, atop a tower in a lithograph of Escher's, shows this triplet of cubes ... but surely, much more startling than this coincidence is the **"Wasserfall"** itself! A simplified version of this *impossible figure*, with water now sealed off in glass tubing, is shown above. Apropos such figures, Penrose observes that a drawing D may not be a perspective representation of any rigid object O even if it is a union of open subsets D_i which are representations of such objects O_i. To get a global obstruction he fixes a point A_{ij} in each nonempty $D_i \cap D_j$ and associates to the directed edge U_iU_j , of the nerve of the covering, the ratio d_i/d_j of the distances to O_i and O_j along the ray through A_{ij} ... however this 1-cochain depends, as such, on the choice of the points A_{ij} ... and is a 1-cocycle, i.e. $d_j/d_k(d_i/d_k)^{-1}(d_i/d_j) = 1$ if $D_i \cap D_j \cap D_k$ is nonempty, only under some extra condition ... then we get a cohomology class in $H^1(D ; R^+)$, etc. ...

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Notes

1. Whenever I revisit logic, this oh so messy, and therefore so alluring, part of our garden, the book that I open first is Manin's, *A Course in Mathematical Logic* (1977). Just its pages 82 to 95 will tell you, with a minimum of fuss, but without pulling any punches either, what ***quantum logic*** is all about: the phrase "startling reason" is on page 84. Besides these quantum roots, this logic owes a lot also, it seems, to a parable -- about a wise man who found a match for his beautiful daughter by asking each suitor to guess which of three closed boxes contained a gem! -- in Specker's, *The logic of non-simultaneously decidable propositions*, a popular paper which appeared in German in 1960.

2. The argument of the opening paragraph showing 'S^2 is startling' used only the intermediate value theorem, an elementary topological result. On page 70 of the 1967 paper by Kochen and Specker (which Manin follows) it is written that this can be proved *"simply either by a direct topological argument or by applying a **theorem of Gleason**."* But I don't think (see below) that the first part of this remark pertains to an argument like ours, and about the theorem which can alternatively be applied, it needs to be stressed that it is *far* deeper. I've barely managed so far to check the steps of the argument presented in Gleason's masterly, *Measures on the closed subspaces of a hilbert space* (1957), and as we all know, merely having checked the steps is by no means the same as having understood a proof.

3. As the second paragraph shows this qualitative version would inevitably lead anyone to the quantitative version with the best possible bound. Which is from **J S Bell**'s beautiful paper, *On the problems of hidden variables in quantum mechanics,* Rev. Mod. Phy. 38 (1966) 447-452**.** The individual algebraical steps in Bell's proof – see section V entitled *Gleason* – are all dead easy, but I really understood it only when, working backwards, I had managed to draw Figure 1 and had obtained the more conceptual qualitative version. As against this, it is clear from some remarks on page 70 of Kochen-Specker that their 'direct topological argument' – whatever it was? – was such that it did not yield a quantitative version. Also, the analogue of Bell's quantitative result which they prove in this paper (equivalently Lemma 12.13 in Manin) only gives us the inferior bound $\sin^{-1}(1/3)$. But we note that this number is equally platonic – $\cos^{-1}(1/3)$ *is the side of a cube*! – and that their quite different and more algebraical argument is equally interesting. They take 2 disjoint orthonormal frames joined by a great circle arc of length $\pi/2$ and add some iterated vector products of these 6 points to obtain a finite set whose perpendicularity graph cannot be coloured in the stipulated way.

4. Though obvious, the fact that his argument also gives finite startling subsets was not explicit in Bell's paper. On the other hand, the paper of Kochen and Specker was focussed on them, especially an almost startling subset with 117 directions. As Mermin's rollicking review, *Hidden variables and the two theorems of John Bell,* Rev. Mod. Phy. 65 (1993) 803-814 points out, their depiction of the perpendicularity graph of this set soon became almost the **logo** of quantum logic! But this excitement is natural : *the moment you have a finite (almost) startling set you can write down a classical tautology which is not a quantum tautology,* etc.

5. I recall that*"Perfectly proportioned"* and *"Beesmukhi"* are the first two lectures in *"213, 16A" and Mathematics*, and its last lecture ***"Magic carpet"*** shows that, *for all non-platonic* {p,q} *also, there is a quantization into finitely many regular* p-*gons,* q *at each vertex, but now of some closed surface covered by the plane, and we are not assuming Euclid's fifth postulate*, i.e., we have generalized his geometry in the obvious way indicated by the intuition that straight lines are circles with radius $c = \infty$. However, *the full classification of these relativistic* (i.e. with $c < \infty$) *tilings and their finite symmetry groups*—a problem that Euclid himself could have posed!—*is still not known* ... Also, besides orthonormal frames, equivalently octahedral or cubical tilings of the 2-sphere, there are ever more startling quantum logics associated to each pair of dual tilings, but as the genus increases, more truth values are needed ...

(to be continued)