WU'S "A THEORY OF IMBEDDING .. " (contd.)

CHAPTER FOUR. We saw that if some *subdivision* of K embeds linearly in $\mathbb{R}^m$ then $o^m(K_*^2) = 0$. This will now be shown to imply the necessary conditions found by **Thom**, in terms of the inverse **Steenrod** operations of K, for the linear embeddability of some simplicial complex *homotopy equivalent* to K in $\mathbb{R}^m$.

For this purpose Wu shows that Steenrod's operations can be defined easily using a generalized Künneth theorem of **Richardson-Smith** which computes the cyclically equivariant cohomology of K^p.

A. **Fixed subcomplex.** We will always assume that a simplicial complex E, on which the action of a finite group G is being considered, is **fine** enough with respect to the group action, i.e. that

$$g.\sigma = \sigma,\ \theta \subseteq \sigma \Rightarrow g.\theta = \theta.$$

This ensures that the *(co)homology of the equivariant (co)chains of* E is is an invariant of the underlying equivariant homotopy type of E.

Also it ensures that the **fixed points** $\{x:\ g.x = x\ \forall\ g \in G\}$ of the group action constitute the space of the subcomplex $F = \{\sigma:\ g.\sigma = \sigma\ \forall\ g \in G\}$.

More generally each normal subgroup of G gives rise to the subcomplex of simplices fixed by it, but if G is *simple*, then the action is *free away from* F, i.e. each point [simplex] of E \ F has $|G|$ distinct [disjoint] conjugates.

In fact, resuming the discussion of Chapter II, which dealt with the fixed point free case $F = \emptyset$, we'll from now on confine ourselves to the finite *abelian* simple, i.e. cyclic prime order, groups $G = \mathbb{Z}_p = \langle T \rangle$.

Proposition 1. *For any* [fine] $\mathbb{Z}_p$*-complex* E, *with fixed subcomplex* F, *the direct sum decomposition*

$$C_*^{d\ \text{or}\ s}(E;\mathbb{F}_p) \cong C_*^{d\ \text{or}\ s}(E,F;\mathbb{F}_p) \oplus C_*(F;\mathbb{F}_p),$$

obtained by splitting any chain $c \in C_*^d(E)$ (resp. $\in C_*^s(E)$) *into the parts* a *and* b **supported** *on* E \ F *and* F, *induces a decomposition*

$$H_*^{d\ \text{or}\ s}(E;\mathbb{F}_p) \cong H_*^{d\ \text{or}\ s}(E,F;\mathbb{F}_p) \oplus H_*(F;\mathbb{F}_p).$$

Proof. Since the action is free on E \ F we can find a chain a' such that $a = s(a')$ (resp. $a = d(a')$).

Now note that s (resp. d) vanishes mod p on simplices of F. So $\partial a = \partial s(a') = s\partial(a')$ (resp. $\partial a = \partial d(a') = d\partial(a')$) must be supported on $E \setminus F$, while ∂b is of course supported on F. So if c is a cycle, i.e. $\partial c = 0$, then both ∂a and ∂b are zero, i.e. both a and b are cycles.

[Above argument does not dualize because δb need not be supported on F.]

One obtains the required homological decomposition because it can be likewise checked that if c is a boundary, then both a and b are boundaries. *q.e.d.*

The above result does not dualize to cohomology: e.g. $H^*_d(E;\mathbb{F}_p)$, which identifies with the cohomology of the space $E/\mathbb{F}_p$, need not be the direct sum of $H^*_d(E,F;\mathbb{F}_p)$ and $H^*(F,\mathbb{F}_p)$, but rather is related to them by the exact sequence of the pair of spaces $(E/\mathbb{F}_p, F)$.

Exact sequences. There is the Richardson-Smith (co)homology sequence of the free relative complex (E,F). Besides there is another one arising from the fact that s (resp. d) applied to the (co)chains of E has as image all the d (resp. s) (co)chains of E vanishing on F. In addition there is the ordinary exact sequence of the pair (E,F), as well as that of its quotient $(E/\mathbb{F}_p, F)$. We omit discussion of the [mostly obvious] maps between these sequences [cf. also **Bott** and **Aeppli-Akiyama**].

Comments

(1) Is there a *nice* [?] generalization of Richardson-Smith theory to all finite [simple] groups G?

[Papers of **Swan** / **Burghelea** suggest that **Tate** / cyclic cohomology might give such a generalization. Also it might be of interest to use as coefficients a *number field* $\mathbb{F}$ whose Galois group over $\mathbb{Q}$ is G ?]

(2) *The bigger the acting group, the more one needs to subdivide before the complex becomes fine.*

For example for the permutation groups 1, $\mathbb{Z}_p$, and S_p on p letters, p a prime, acting on the p-fold join of K by permuting factors, the simplicial complex $K^{\mathbf{p}}$, its *Wu subdivision* $W(K^{\mathbf{p}})$, and its *Bier subdivision* $W(K^{\mathbf{p}})$, are, respectively, fine enough with respect to the group action.

Thus if we restrict to the groups $\mathbb{Z}_p$, p a prime, there is no point in using the finer Bier subdivision, which will come into play for other permutation groups.

[In this context cyclic semi-simplicial complexes serve to make even the action of the *infinite* group of circular rotations fine!]

(3) *Can the Steenrod operations be defined canonically in the Wu*

subdivision ?

Given a degree q cocycle u of K we have a *non-homogenous* cochain Pu of $W(K^p)$ defined by $(Pu)(\theta.(\sigma_1, .. ,\sigma_p)) = u(\theta\cup\sigma_1). .. .u(\theta\cup\sigma_p) \in \mathbb{F}_p$. Note that the right side can be nonzero only if $|\theta\cup\sigma_1| = .. = |\theta\cup\sigma_p| = q$, so we have $\deg(\theta.(\sigma_1, .. ,\sigma_p)) = q \le |\theta| + |\sigma_1| + .. + |\sigma_p| \le qp$. Thus Pu lives in degrees q through qp.

Though Pu *is* $\mathbb{Z}_p$ *equivariant it is not a cocycle of* $W(K^p)$. For example in degree q, i.e. when $\sigma_1 = .. = \sigma_p = \emptyset$, we have $(Pu)(\theta) = u(\theta)$ [because $x^p = x \;\forall\; x \in \mathbb{F}_p$], which obviously need not be a cocycle *of* $W(K^p)$, even though it is of the diagonal, and this restriction to the diagonal gives the identity map $[u] \mapsto [u]$ of $H^*(K)$.

The pth cup power seems to be definable from Pu. To do this we work in dimension p(q-1), i.e. in degree t = p(q-1)+1. The point to note is that the *degree* t *part of* Pu *is a cocycle of a subcomplex containing the diagonal*, viz. the the cell subcomplex K^p. The cohomological restriction to the diagonal now gives $[u]^p$.

It seems that a similar modified [?] *cohomological restriction to the diagonal is possible for all of* Pu *and will define the entire Steenrod class* $P[u] = [u] + \mathcal{P}^1[u] + \mathcal{P}^2[u] \;...\; + [u]^p$. If so this would be really nice, because the Wu subdivision itself would have served to spread the p fold join of u into lower dimensions rather than Smith morphisms or Borel trick etc.

FADELL-NEUWIRTH

Throughout M will denote a manifold of dimension ≥ 2, and M \ {k pts} will also be denoted M_{-k}.

(A) *Let* $F_n(M) = \{(x_1, ... ,x_n) \in M \times .. \times M : x_i \ne x_j \;\forall\; i \ne j\}$ *be the* **nth configuration space** *of* M. *Then the* **first coordinate map** $F_n(M) \longrightarrow M$, $(x_1, ... ,x_n) \mapsto x_1$, *is a fibration with fiber* $F_{n-1}(M_{-1})$.

We omit the easy verification.

(B) *If a fibration* $F \subseteq E \rightarrow B$ *admits a section, then*

$$\pi_i(E) = \pi_i(B) \oplus \pi_i(F) \;\forall\;,\; i \ge 2.$$

Proof. This follows because the homotopy sequence splits into short exact sequences. *q.e.d.*

(C) Theorem. *If first coordinate map* $F_n(M) \to M$ *has a section, then*

$$\pi_i(F_n(M)) = \oplus_{0\le k<n}\, \pi_i(M \setminus k \text{ pts}),\ i \ge 2.$$

So, for example,

$$\pi_i(F_n(\mathbb{R}^m)) \cong \oplus_{1\le k<n}\, \pi_i(S^{m-1} \vee \ldots \vee S^{m-1} \text{ (k times)}),\ i \ge 2.$$

Proof. The point to note is that the first coordinate map $F_{n-1}(M_{-1}) \to M_{-1}$, which has $F_{n-2}(M_{-2})$ as fiber, *always* has a section.

This follows because we can choose the second, third, etc. coordinates of the section to contract along different directions of a neighbourhood of the missing point $M \setminus M_{-1}$: then they are all distinct from each other, and also from the first coordinate, which is identity.

Now iterate the construction, and use (B) starting with the last fibration $F_2(M_{-n+2}) \to M_{-n+2}$, which has fiber $F_1(M_{-n+1}) = M_{-n+1}$.

The second part follows because m-space is contractible, and if we omit k points from it we get the homotopy type of a bouquet of k spheres of dimension m-1. *q.e.d.*

(D) Existence of section of $F_n(M) \to M$. This follows easily if M has an identically nonzero vector field i.e., **zero Euler characteristic** [so e.g. they also get a bouquet formula similar to above for $\pi_i(F_n(S^m))$, m odd], and also if there is some [not necessarily deformation] **retraction** $M \to L$, with manifold L having a section of above kind.

This establishes existence of section for lots of manifolds because **any closed manifold M with first Betti number nonzero retracts to a circle** [they cite **Whyburn**'s book for this], so such M's have sections of above kind.

On the other hand they check that for n ≥ 3, the nth configuration space of an even-dimensional sphere has no section, and neither do the configuration spaces of manifolds having the **fixed point property** have sections.

[Note that, as against $\mathbb{R}^m$, an m-ball B^m has the fixed point property. But not being a manifold without boundary, above theory does'nt apply to it: the topological type of the space obtained by omitting some points depends on these points.]

(E) The nth configuration space is the quotient of the identity component of the **group of homeomorphisms** of M by the subgroup keeping each of some n chosen points fixed. Using the associated exact homotopy sequence the authors indicate some situations in which the homotopy groups of this subgroup are same as of the full homeomorphism group.

Also they make some computations of the **braid group** of M, i.e. the fundamental group of $F_n(M)/\Sigma_n$, where Σ_n is the nth symmetric group.

Comments

(1) Remember that $H_i(F \times B) \cong \oplus_{j+k=i} H_j(F) \otimes H_k(B)$ [with field coefficients] as against $\pi_i(F \times B) \cong \pi_i(F) \oplus \pi_i(B)$.

[Likewise homology groups of disjoint or one-point unions are given by rules quite different from those for homotopy groups.]

(2) **So the formula for $\pi_i(F_n(\mathbb{R}^m))$ does not suggest that $F_n(\mathbb{R}^m)$ has the homotopy type of an (m-1)-complex**, but rather of the [possibly slightly twisted] product of n-1 complexes, each of which is an (m-1)-complex.

[Most probably available computations will show that the homology of $F_n(M)$ can be nonzero in a dimension going to infinity with n ?]

(3) It would be interesting nevertheless to find a definition, which associates to each simplicial complex K, another simplicial complex $F_{n^*}(K)$ [say of dimension n.dimK] which has always the homotopy type of the nth configuration space $F_n(X)$, where $X = |K|$.

(4) Note that for the **nth join configuration** $\mathbb{F}_n(X)$, $X = |K|$, we have solved the problem analogous to (3): *its homotopy type coincides with that of the subcomplex of the* k*-fold join of* K *determined by pairwise disjoint sequences of simplices of* K.

So, **as against $F_n(X)$, the homotopy type of $\mathbb{F}_n(X)$ gives information about the number N of vertices required to triangulate X**: it must be more than the dimension till which its homology is nonzero.

(5) While looking at braid groups the authors use a result of **Smith** to the effect that if, for a *finite* dimensional complex, π_i's are all zero for $i \geq 2$, then π_1 has no elements of finite order.

SEGAL'S PAPER ON CONFIGURATION SPACES

As is usual with his papers, this paper of Segal is elegant, well-written and quite informative.

Throughout the following S and Ω will denote the reduced **suspension** and **loop** functors in the category of pointed spaces.

(A) A **partial monoid** is a pointed set [or space] (X,1) equipped with a partially defined [continuous] multiplication $X \times X \longrightarrow X$ for which x.1

$= x = 1.x \; \forall \; x$, and $(x.y).z = x.(y.z)$ whenever the two sides are defined. It gives rise to the following "Hochschild" *semi-simplicial complex* which will also be denoted by X.

The *n-simplices* are length n multipliable sequences of points of X, the *face operators* are obtained by multiplying two consecutive entries or omitting the last one, and the *degeneracy operators* are obtained by insertions of 1.

The *geometrical realization* of this semi-simplicial complex will be denoted by BX.

(B) For any pointed space (X,0), let $C_n(X)$ be the **configuration space** C_n of finite subsets α of $\mathbb{R}^n$, **labelled** by points of X, with two labelled sets identified iff their nonzero labelling is same.

Note that union of two such *disjoint* labelled finite sets α and β is also a labelled finite set, and that under this multiplication $C_n(X)$ is a partial monoid. The main result of the paper is the following.

Segal's Theorem. $BC_{n-1}(X) \cong C_{n-1}(SX) \simeq \Omega^{n-1}(S^n X)$.

(C) **Corollaries and Remarks.**

(i) Note that $C_0(X)$ is the **free monoid** MX on X, so above result generalizes **James' Theorem**: $BM(X) \simeq S(X)$.

(ii) Also, if X is *path connected*, then $\Omega BC_{n-1}(X) \simeq C_n(X)$, so for such an X one obtains **May's Theorem**: $C_n(X) \simeq \Omega^n S^n(X)$.

(iii) This is not true when X is not path connected, but even now one has at least, for each k, $H_k(C_n(X)) \cong H_k(\Omega^n S^n(X))$, provided n is large enough.

(iv) Segal also gives a picturesque discription of a map $E\colon C_n \to \Omega^n S^n$ which induces the above homology isomorphism *for the case* $X = S^0 = \{0, \pm 1\}$: place at each point of $\alpha \subset \mathbb{R}^n$ the **unit positive charge** +1, then

$$E(\alpha)\colon (\mathbb{R}^n \cup \infty, \infty) \to (\mathbb{R}^n \cup \infty, 0),$$

is the **electrostatic field** of this charge distribution.

(D) **Quillen's subdivision**. If Δ^n is the closed n-simplex on $\mathbf{n} = \{0, 1, \ldots, n\}$ and ij, $i \le j$, denotes the barycentre of $\{i,j\}$, then the simplicial complex consisting of all simplices of the type,

$$\{i_1 j_1, i_2 j_2, \ldots, i_k j_k\},\ i_1 \ge i_2 \ge \ldots \ge i_k,\ j_1 \le j_2 \le \ldots \le j_k,$$

constitutes a sibdivision $Q(\Delta^n)$ of Δ^n.

Note that each order preserving map **n** $\rightarrow$ **m** induces a simplicial map $\Delta^n \rightarrow \Delta^m$. This enables us to associate functorially to every semi-simplicial complex A, a semi-simplicial complex Q(A), such that the realization of Q(A) is a subdivision of the realization of A.

Note that the set of "edges" of A coincides with that of the "vertices" of Q(A), and likewise more generally the set of degree d simplices of the s.s.c. Q( ) coincides with that of the degree 2d simplices of A.

Comments

(1) Any spce X defines a semi-simplicial complex *in the original sense of Eilenb g-Zilber* [i.e.without degeneracies] as follows: the *n-simplices* are length n+1 sequences of points of X and *face maps* are given by oission of a term.

The infine join X·X· ... is the geometrical realization of the above semi-simpcial complex.

This fol ws because the space of n-simplices is the (k+1)-*fold product* X^{k+1} andthe realization is the quotient of the disjoint union $X^{k+1} \times \Delta^k$ under th identifications dictated by the face maps.

Likewis, if we limit ourselves to length p sequences only, then we'll obtain the **pfold join** X· ... ·X.

And, i X i *pointed*, then we also have degeneracies given by insertions of th bas point, so we can speak likewise of infinite or p-fold **reduc d jois** of X.

(2) It is atural to ask if the (co)homology of the infinite join X·X· ... [and kewise for p-fold joins etc.] can be calculated from the (co)chain mplex of the above semi-simplicial complex ?

In fact thaforementioned (co)chain complex seems to be the [acyclic ?] complex o *Alexander-Kolmogorov (co)chains without the localization condition* his condition corresponds to restricting to the *diagonal* of ... join a as is well-known gives the (co)homology of X] ?

Obviously iis train of ideas should also be close to the **Dold-Thom Theorem** oinfinite symmetric products ?

(3) *Quill subdivisions are analogous to Wu subdivisions*, because in both one es compatible subdivisions of the standard simplices to subdivide e spaces in question.

Quillen's bdivisions Q(A) *are more economical than barycentric subdivisic* B(A) [the former being a stellar subdivision in which just the edgesre derived] in the sense that new number of simplices is lesser. I note that any map **n** $\rightarrow$ **m** $\Delta^n \rightarrow$