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POSITIVELY CURVED BILINEAR FORMS

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ABSTRACT. A characterisation of the positivity of sectional curvature is given in terms of the angular spread of the second fundamental form.

1. Introduction. We will deal only with real and finite-dimensional vector spaces; occasionally their dimensions will be denoted by superscripts. Our principal objects of study are vector-valued bilinear forms $B: V^n \times V^n \to W^p$, with special emphasis on their properties vis-à-vis a given inner product $\langle\ \rangle$ on W. In this context Weinstein [3] has considered the *associated curvature form* $R_B: \Lambda^2 V \times \Lambda^2 V \to \mathbf{R}$ defined by

$$(1)\qquad R_B(v_1 \wedge v_2, v_3 \wedge v_4) = \langle B(v_1, v_3), B(v_2, v_4)\rangle - \langle B(v_1, v_4), B(v_2, v_3)\rangle.$$

(The reason for this terminology comes from differential geometry: If V (resp. W) is a tangent (resp. normal) space of an n-manifold M^n isometrically immersed in $\mathbf{R}^{n+p}$ and $B: V \times V \to W$ is the second fundamental form of the immersion, then the curvature R of M^n is related to B by (1).) We will say that B is *positively curved* if $R_B(\omega, \omega) > 0$ for all nonzero decomposable elements ω of $\Lambda^2 V$. The set $C_B = \{B(x, x) | x \neq 0\}$ will be called the *cone* of B. After giving some preliminary refinements of Weinstein's methods we prove in §2 that *a symmetric bilinear form $B: V^n \times V^n \to W^p$, $\langle\ \rangle$ is positively curved if and only if the restriction of B to each 2-dimensional subspace has a cone which can be contained in the interior of some 3-dimensional orthant of the inner product space W^p, $\langle\ \rangle$* (see (2.3.4); also (2.2.7) and (2.3.3) for more quantitative aspects of the result). §2 also contains some sufficient conditions under which R_B is *positive definite*, i.e. $R_B(\omega, \omega) > 0$ for all nonzero $\omega \in \Lambda^2 V$. (It is known (see e.g. [**2**, pp. 364–368]) that a closed Riemannian manifold M whose curvature R is positive definite at all points must satisfy some strong topological conditions; relatively much less is known about positively curved closed manifolds.) If $n \leq 3$ all elements of $\Lambda^2 V$ are decomposable and thus B is positively curved iff R_B is positive definite; Weinstein [**3**] (or (2.2.5) below) proved that this remains true for all n provided $p \leq 2$. In §3 we give an example of *a positively curved $B: V^4 \times V^4 \to W^3$, $\langle\ \rangle$ for which R_B is not positive definite.*

2. Bilinear forms $B: V^n \times V^n \to W^p$.

(2.1) A symmetric bilinear form $B: V \times V \to W$ determines a *quadratic function* $B: V \to W$ by $B(x) = B(x, x)$.

(2.1.1) *A continuous map $B: V \to W$ is a quadratic function of some symmetric bilinear form iff it obeys the parallelogram law $B(x+y) + B(x-y) = 2B(x) + 2B(y)$.* A proof of this result, for the scalar case $p = 1$, can be found e.g. in [**1**, pp. 245–246]; the same argument works for all p.

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The subset $B(V^n - \{0\}) = \{B(x) | x \neq 0\}$ of W^p shall be called the *cone* of B and denoted by C_B; note that if $\lambda > 0$ and $w \in C_B$ then $\lambda w \in C_B$.

A bilinear form $B: V^n \times V^n \to W^p$ shall be called *diagonalisable* if V^n admits a basis $v_1, v_2, \ldots, v_n$ such that $B(v_i, v_j) = 0$ whenever $i \neq j$.

(2.1.2) *For $n \geq 3$ and $p = 2$ a bilinear form B with $0 \notin C_B$ is diagonalisable.* To see this choose any basis w_1, w_2 of W^2 and let $B(x,y) = B_1(x,y)w_1 + B_2(x,y)w_2$. Diagonalisability of B is equivalent to the simultaneous diagonalisability of the two $\mathbf{R}$-valued symmetric bilinear forms B_1 and B_2. It is known (see e.g. [**1**, p. 256]) that this is possible under the given conditions $n \geq 3$ and $B_1(x)^2 + B_2(x)^2 \neq 0$ for all $x \neq 0$.

If $w_1, w_2, \ldots, w_k$ are nonzero elements of W^p, then $C(w_1, w_2, \ldots, w_k) \subseteq W^p$ will denote the set of all points $\sum \lambda_i w_i$, $\lambda_i \geq 0$, $\sum \lambda_i > 0$. If v_i is a diagonalising basis and $\lambda_i \geq 0$ then $\sum \lambda_i B(v_i) = B(\sum \sqrt{\lambda_i} v_i)$ which implies $C_B = C(w_1, w_2, \ldots, w_n)$ where $w_i = B(v_i)$. For $p \geq 3$ it is easy to give examples of bilinear forms B with $0 \notin C_B$ whose cone is not of the type $C(w_1, \ldots, w_k)$; thus (2.1.2) does not generalize to $p \geq 3$.

(2.1.3) *For $n \geq 3$ and $p = 2$, $0 \notin C_B$ only if C_B is of the type $C(w_1, w_2)$. For $n = p = 2$, $0 \notin C_B$ only if $C_B = W^2 - \{0\}$ or $C_B = C(w_1, w_2)$ for some w_1, w_2.* (For $n = 1$, $0 \notin C_B$ iff $C_B = C(w)$ for some w; from now on we ignore this trivial case and take $n \geq 2$.) The first part follows from (2.1.2) and the above remark which show that a C_B not containing the origin is made up of all open half rays from the origin which pass through a polygonal region $\operatorname{conv}(w_1, \ldots, w_n) \subseteq W^2 - \{0\}$; thus with a proper choice of i and j we must have $C_B = C(w_i, w_j)$. To see the second part we choose a basis v_1, v_2 of V^2 and put $w_1 = B(v_1)$, $w_2 = B(v_2)$ and $w_3 = B(v_1, v_2)$. The ellipse $E \subseteq V^2$ consisting of all points $\cos\theta v_1 + \sin\theta v_2$ is mapped by $B: V^2 \to W^2$ onto the set $E' \subseteq W^2$ consisting of points $\cos^2\theta w_1 + \sin^2\theta w_2 + 2\sin\theta\cos\theta w_3 = (w_1 + w_2)/2 + \cos 2\theta((w_1 - w_2)/2) + \sin 2\theta w_3$. Clearly E' is a point, a closed interval or an ellipse; further if $0 \notin C_B$, E' is contained in $W^2 - \{0\}$. Our cone C_B is made up of all open half rays from the origin which pass through E'. Hence $C_B = W^2 - \{0\}$ or $C_B = C(w_1, w_2)$ depending on whether or not 0 lies in the convex hull of E'.

(2.2) We now turn to the properties of B vis-à-vis some inner products. Whenever W^p is equipped with an inner product we will define the curvature form $R_B: \Lambda^2 V \times \Lambda^2 V \to \mathbf{R}$ as in §1. Occasionally (e.g. in the differential geometric situation alluded to in §1) V^n also comes equipped with an inner product. Then we will also consider the *Ricci curvature* $S_B: V \times V \to \mathbf{R}$ which is defined by contracting R_B, i.e.

$$S_B(x,y) = \sum_i R_B(x \wedge v_i, y \wedge v_i), \tag{2}$$

where v_i is any orthonormal basis of V^n. Note that R_B positive definite implies B positively curved which in turn implies S_B positive definite.

We define the *angle* between two nonzero vectors w_1, w_2 of an inner product space W to be the number in $[0, \pi]$ whose cosine equals $\langle w_1, w_2 \rangle / (|w_1|\,|w_2|)$; further we define the *angular diameter* of a set A contained in some open half space of W to be the supremum of the angle between any two elements of A.

(2.2.1) *If Ricci curvature S_B is positive definite, then $0 \notin C_B$. If B is positively curved, then C_B is contained in some open half space of W^p and angular diameter* *of C_B is less than $\pi/2$.* The first part follows by noting that (1) and (2) imply $S_B(x,x) = \sum_i \langle B(x), B(v_i)\rangle - \sum_i \langle B(x,v_i), B(x,v_i)\rangle$; this expression can be positive only if $B(x) \neq 0$. For the second part take any two points w_1, w_2 of V_B which are not on the same half ray from the origin, and let $B(v_1) = w_1$, $B(v_2) = w_2$. Then v_1, v_2 are linearly independent; so $R_B(v_1 \wedge v_2, v_1 \wedge v_2) = \langle w_1, w_2\rangle - \langle B(v_1,v_2), B(v_1,v_2)\rangle$ is positive, which can happen only if $\langle w_1, w_2\rangle > 0$. Thus C_B lies in an open half space and has angular diameter less than $\pi/2$.

(2.2.2) *If $f: M^n \to \mathbf{R}^{n+2}$, $n \geq 3$, is an isometric immersion of an orientable Riemannian manifold M^n with Ricci curvature positive definite at all points, then the normal bundle of f must be trivial.* (This means that the tangent bundle of M^n is stably trivial and thus, if M^n is compact, the Pontryagin and Stiefel-Whitney classes of M^n vanish.) To see this note first that a 'clockwise' sense can be prescribed in a continuous way to the normal spaces W^2. But by (2.2.1) and (2.1.3) C_B is of type $C(w_1, w_2)$. The unit normal vector w_a which bisects this sector and the unit vector w_b obtained by rotating w_a clockwise through angle $\pi/2$ now give us the required continuous trivialization of the normal spaces W^2. (The same argument works also for $n \geq 2$ provided one assumes that M^n is positively curved; this yields a result of Weinstein [**3**].)

(2.2.3) *For a diagonalisable B* (a) *R_B is positive definite iff* (b) *B is positively curved iff* (c) *angular diameter of C_B is less than $\pi/2$.*

(a) $\Rightarrow$ (b) is trivial while (b) $\Rightarrow$ (c) is in (2.2.1). To prove (c) $\Rightarrow$ (a) let $v_1, v_2, \ldots, v_n$ be a diagonalising basis of V^n and let $B(v_i) = w_i$; we are given that $\langle w_i, w_j\rangle$ is positive for all i, j. We note that if $i < j$ and $k < l$, then $R_B(v_i \wedge v_j, v_k \wedge v_l) = \langle B(v_i, v_k), B(v_j, v_l)\rangle - \langle B(v_i, v_l), B(v_j, v_k)\rangle$ is zero unless $i = k$ and $j = l$; therefore, $v_i \wedge v_j$, $i < j$, constitute a diagonalising basis for $R_B: \Lambda^2 V \times \Lambda^2 V \to \mathbf{R}$. Also each diagonal value $R_B(v_i \wedge v_j, v_i \wedge v_j) = \langle w_i, w_j\rangle$ is positive. Therefore, $R_B(\omega, \omega) > 0$ for all nonzero elements ω of $\Lambda^2 V$.

A subset of W^p will be called a k-dimensional *orthant* if for some orthonormal set of k elements $\{w_1, \ldots, w_k\} \subseteq W^p$ it equals $C(w_1, w_2, \ldots, w_k)$; a 2-orthant (resp. a 3-orthant) will also be called a *quadrant* (resp. an *octant*). The following observation is due to Weinstein [**3**].

(2.2.4) *If C_B is contained in the interior of some p-orthant of W^p, then R_B is positive definite.* Choose an orthonormal $\{w_1, \ldots, w_p\} \subseteq W^p$ such that $C_B \subseteq \operatorname{int} C(w_1, \ldots, w_p)$. For $1 \leq i \leq p$ define bilinear forms $B_i: V^n \times V^n \to \mathbf{R}$ by $B(x,y) = \sum_i B_i(x,y) w_i$; since $B_i(x,x) = \langle B(x,x), w_i\rangle$ is positive whenever $x \neq 0$ we see that each B_i is positive definite. Since B_i is diagonalisable (c) $\Rightarrow$ (a) of (2.2.3) shows that R_{B_i} is positive definite. But $R_B = \sum_i R_{B_i}$ by a simple calculation. Hence R_B is positive definite.

It is not hard to give an example of a diagonalisable $B: V^n \times V^n \to W^3, \langle\,\rangle$ whose C_B has an angular diameter less than $\pi/2$ but cannot be contained in the interior of any octant. Therefore, by using (2.2.3), we see that the *converse of* (2.2.4) *is false.* However, for $p = 2$ it is clear that a set $A \subseteq W^2$ with angular diameter less than $\pi/2$ is contained in the interior of some quadrant. This yields the following lemma of Weinstein [**3**].

(2.2.5) *For $p = 2$,* (a) *R_B is positive definite iff* (b) *B is positively curved iff* (c) *angular diameter of C_B is less than $\pi/2$ iff* (d) *C_B is contained in the interior of some quadrant.*

(c) $\Rightarrow$ (d) is the remark just made. The implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a) are valid for all p, the first being trivial and the other two being in (2.2.1) and (2.2.4), respectively.

One can formulate a *generalized Cauchy-Schwarz inequality* as follows: "$\langle B(x), B(y)\rangle > 0$ for all nonzero x and y iff $\langle B(x), B(y)\rangle > \langle B(x,y), B(x,y)\rangle$ for all linearly independent x and y". (The classical inequality is the case $p = 1$ of this assertion: one is looking at bilinear forms $B: V^n \times V^n \to \mathbf{R}$, $\langle\ \rangle$ where $\langle\ \rangle$ is the ordinary multiplication in $\mathbf{R}$.) (2.2.5) (c) $\Rightarrow$ (b) says in fact that the generalized Cauchy-Schwarz inequality is true for $p = 2$; we will see in (2.3) that it is false for $p = 3$.

(2.2.6) *If angular diameter of C_B is less than $\cos^{-1}\sqrt{(p-1)/p}$, then C_B can be contained in the interior of some p-orthant of W^p.* In fact we can show that C_B is in the interior of any orthant $C(w_1, \ldots, w_p)$ for which $a = (w_1 + \cdots + w_p)/\sqrt{p}$ is in C_B. To see this we check that the maxima of the function $\langle a, u\rangle$ as u runs over all unit vectors in the boundary of the orthant (such a u is of the type $\sum u_i w_i$, $u_i \geq 0$, $\sum u_i^2 = 1$ with at least one $u_i = 0$) is $\sqrt{(p-1)/p}$ and is attained at the p values $u^i = (w_1 + \cdots + \hat{w}_i + \cdots + w_p)/\sqrt{p-1}$. Thus all open half rays from the origin which make an angle of less than $\cos^{-1}\sqrt{(p-1)/p}$ with a—and so a fortiori all of C_B—are contained in the interior of the orthant.

If one has an isometric immersion $f: M^n \to \mathbf{R}^{n+p}$ whose second fundamental form has a cone C_B of angular diameter $< \cos^{-1}\sqrt{(p-1)/p}$ at all points, then the curvature tensor R of M^n must be positive definite at all points. (This in turn has the usual interesting topological consequences, e.g. for n even and M^n compact, the Euler characteristic of M^n must be positive.) This follows by (2.2.4) and (2.2.6).

We say that a cone C_B lying in an open half space of W^p has *central symmetry* if there exists a half ray (an "axis" of C_B) whose angular distance from any point of C_B is at most one half the angular diameter of C_B. *For a centrally symmetric C_B the bound of* (2.2.6) *can be improved to* $2\cos^{-1}\sqrt{(p-1)/p}$; this follows by the same argument taking care this time to choose the a along the axis of C_B. Note that by (2.1.3) one has central symmetry for $p = 2$; this gives us the bound $2\cos^{-1}\sqrt{1/2} = \pi/2$ which is best possible by (2.2.5). We remark that one has central symmetry also in the case $n = 2$, $p = 3$; this will follow from (2.3.1).

(2.2.7) *If angular diameter of C_B is less than $2\cos^{-1}\sqrt{2/3}$, then B is positively curved.* For each 2-dimensional subspace ν of V^n let ω denote the linear span of the image of $\nu \times \nu$ under $B: V^n \times V^n \to W^p$; we equip $\omega \subseteq W^p$ with the induced inner product $\langle\ \rangle$. It is clear from the definition given in §1 that $B: V^n \times V^n \to W^p$, $\langle\ \rangle$ *is positively curved iff its restriction $\beta: \nu \times \nu \to \omega$, $\langle\ \rangle$ to each 2-dimensional subspace ν is positively curved.* Since each ω is at most 3-dimensional and since we are given that the diameter of each C_β is less than $2\cos^{-1}\sqrt{2/3}$, it follows from (2.2.4) and the above remark that each β is indeed positively curved.

(2.3) In this section we consider the case $n = 2$, $p = 3$ and give a characterisation of all positively curved $B: V^2 \times V^2 \to W^3$, $\langle\ \rangle$; (2.2.7) shows that this would suffice to characterise all positively curved bilinear forms. A $B: V^2 \times V^2 \to W^3$ will be called *nondegenerate* if its image is not contained in any proper subspace of W^3.

(2.3.1) *For a nondegenerate $B: V^2 \times V^2 \to W^3$, C_B is an elliptical cone contained in an open half space of W^3.* Since B is nondegenerate we must have $0 \notin C_B$. Now, as in (2.1.3), let v_1, v_2 be a basis of V^2 and note that the ellipse $E \subseteq V^2$ consisting of all points $\cos\theta v_1+\sin\theta v_2$ is mapped by $B\colon V^2\to W^3$ to the nondegenerate ellipse $E'\subseteq W^3-\{0\}$ consisting of all points $(w_1+w_2)/2+\cos 2\theta((w_1-w_2)/2)+\sin 2\theta w_3$ where $w_1=B(v_1)$, $w_2=B(v_2)$ and $w_3=B(v_1,v_2)$; C_B consists of all open half rays from the origin which pass through E'. Since B is nondegenerate, the plane of E' does not contain 0 and C_B lies in an open half space of W^3. (Note that $V^2-\{0\}$ is a 2-fold cover of C_B under $B\colon V^2-\{0\}\to C_B$.)

We now consider the metrical properties of this elliptical cone $C_B\subseteq W^3, \langle\ \rangle$. As usual the *axis* of C_B (or of B) will be the open half ray from the origin which passes through the center of mass of the solid angle determined by C_B and the *major and minor semiangles* β_1,β_2 of C_B (or of B) will be the angles subtended at the vertex by the major and minor axes of any section of C_B normal to its axis.

(2.3.2) *If $B\colon V^2\times V^2\to W^3, \langle\ \rangle$ is nondegenerate with major and minor semiangles β_1 and β_2, then we can choose a basis v_1,v_2 of V^2 and an orthonormal basis w_1,w_2,w_3 of W^3 such that*

$$(3)\qquad \begin{aligned} B(v_1)&=\sin\beta_1 w_1+\cos\beta_1 w_2,\\ B(v_2)&=-\sin\beta_1 w_1+\cos\beta_1 w_2 \quad\text{and}\\ B(v_1,v_2)&=\cos\beta_1\tan\beta_2 w_2. \end{aligned}$$

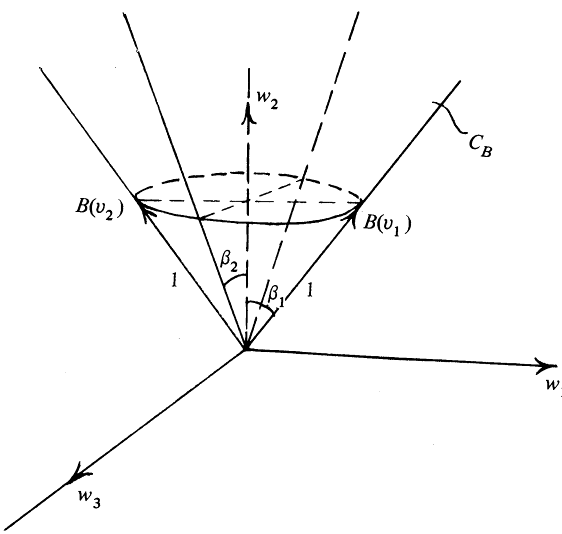

FIGURE 1

Let w_2 be the unit vector along the axis of C_B and let w_1 and w_3 be parallel to the major and minor axes of the normal sections (i.e. the sections normal to the axis) of C_B. Choose v_1 and v_2 so that the first two of equations (3) hold; clearly such a pair v_1,v_2 is linearly independent. We assert that $B(v_1,v_2)=\pm\cos\beta_1\tan\beta_2 w_3$; this suffices to prove the above result because if need be we can replace w_3 by $-w_3$. Just as in (2.3.1) we note that the ellipse $E\subseteq V^2$ consisting of all points $\cos\theta v_1+\sin\theta v_2$ is mapped by $B\colon V^2\to W^3$ to the ellipse $E'\subseteq C_B$ consisting of all points $\cos\beta_1 w_2+\cos 2\theta\sin\beta_1 w_1+\sin 2\theta B(v_1,v_2)$. This ellipse E' has center $\cos\beta_1 w_2$ on the axis of C_B. But only the normal sections of C_B have their centers on the axis. So E' must be the normal section shown in Figure 1 and we must have $B(v_1, v_2) = \pm\cos\beta_1 \tan\beta_2 w_3$.

Given an elliptical cone C lying in an open half space of W^3, we can thus find a nondegenerate form $B: V^2 \times V^2 \to W^3$ whose cone is C; (2.3.2) shows that two nondegenerate bilinear forms B_1 and B_2 have congruent cones C_{B_1} and C_{B_2} iff B_1 is *equivalent* to B_2 in the sense that there exists a linear transformation $f: V^2 \overset{\cong}{\to} V^2$ and an orthogonal transformation $g: W^3, \langle\ \rangle \to W^3, \langle\ \rangle$ such that $B_2 = g \circ B_1 \circ f$.

(2.3.3) *A nondegenerate* $B: V^2 \times V^2 \to W^3, \langle\ \rangle$ *with major and minor semiangles* β_1 *and* β_2 *is* (a) *positively curved iff* (b) $\tan^2\beta_1 + \tan^2\beta_2 < 1$ *iff* (c) C_B *is contained in the interior of some octant.*

Choose bases v_1, v_2 of V^2 and w_1, w_2, w_3 of W^3 as in (2.3.2). By (1) and (3),

$$\begin{aligned} R_B(v_1 \wedge v_2, v_1 \wedge v_2) &= \langle B(v_1), B(v_2)\rangle - \langle B(v_1, v_2), B(v_1, v_2)\rangle \\ &= -\sin^2\beta_1 + \cos^2\beta_1 - \cos^2\beta_1 \tan^2\beta_2 \\ &= \cos^2\beta_1(1 - \tan^2\beta_1 - \tan^2\beta_2), \end{aligned}$$

which is positive only if $\tan^2\beta_1 + \tan^2\beta_2 < 1$. This shows (a) $\Rightarrow$ (b). Since (2.2.4) gives (c) $\Rightarrow$ (a) the proof will be complete once we have shown (b) $\Rightarrow$ (c).

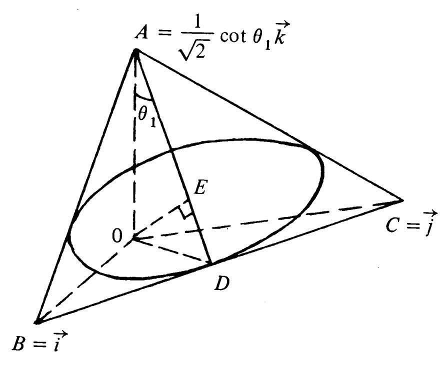

FIGURE 2a

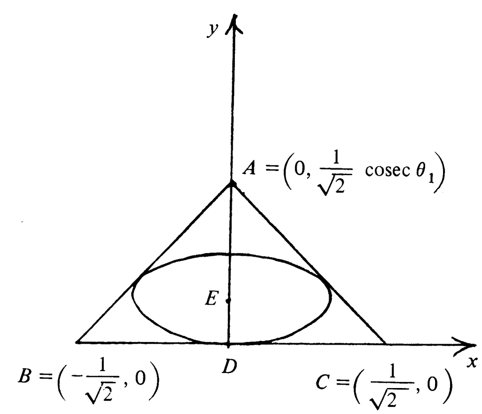

FIGURE 2b

Let $\vec{i}, \vec{j}, \vec{k}$ be any orthonormal basis of W^3 and let $\theta_1 \in (0, \pi/2)$. We denote by ABC the triangle with vertices at $\vec{i}, \vec{j}$ and $(1/\sqrt{2})\cot\theta_1 \vec{k}$ (see Figure 2a). We join A to midpoint D of BC and let E be the foot of the perpendicular from O to AD; since $\angle OAD = \theta_1$ and triangles OAD and EOD are similar, we have $\angle EOD = \theta_1$. In the plane of ΔABC we draw the ellipse $\mathcal{E}$ with center E and with ED as one of its semiaxes and which is tangent to the sides AB and AC. Let θ_2 be the angle subtended by the other semiaxis at O; we assert that $\tan^2\theta_1 + \tan^2\theta_2 = 1$. This assertion suffices to prove (b) $\Rightarrow$ (c) because by choosing θ_1 to be bigger than β_1 by only a small amount we can ensure, by virtue of $\tan^2\beta_1 + \tan^2\beta_2 < 1$, that θ_2 is also bigger than β_2; then C_B would be congruent to an elliptical cone lying in $\operatorname{int} C(\vec{i}, \vec{j}, \vec{k})$ and cutting ABC in an ellipse concentric with and 'parallel' to $\mathcal{E}$.

We choose rectangular axes x and y in the plane of ΔABC as in Figure 2b. The semiaxes of $\mathcal{E}$ being $(1/\sqrt{2})\cos\theta_1\tan\theta_2$ and $(1/\sqrt{2})\sin\theta_1$, its equation is

$$\text{(4)}\qquad \frac{x^2}{\frac{1}{2}\cos^2\theta_1\tan^2\theta_2}+\frac{(y-(1/\sqrt{2})\sin\theta_1)^2}{\frac{1}{2}\sin^2\theta_1}=1.$$

On the other hand the straight line AC has equation $x = 1/\sqrt{2} - \sin\theta_1 y$; making this substitution in (4) we get

$$\text{(5)}\qquad \frac{(1-\sqrt{2}y\sin\theta_1)^2}{\cos^2\theta_1\tan^2\theta_2}+\frac{(\sqrt{2}y-\sin\theta_1)^2}{\sin^2\theta_1}=1$$

which simplifies to

$$\text{(6)}\quad (\sin^4\theta_1+\cos^2\theta_1\tan^2\theta_2)Y^2-(2\sin^3\theta_1+2\sin\theta_1\cos^2\theta_1\tan^2\theta_2)Y+\sin^2\theta_1=0,$$

where $Y = \sqrt{2}y$. Since this quadratic has only one real solution its discriminant is zero. A short calculation shows that the discriminant of (6) equals

$$4\sin^2\theta_1\cos^4\theta_1\tan^2\theta_2(\tan^2\theta_1+\tan^2\theta_2-1).$$

So $\tan^2\theta_1+\tan^2\theta_2=1$.

By applying (2.3.3) to a B such that $C_B = C$ we see that $\tan^2\beta_1+\tan^2\beta_2<1$ is a n.a.s.c. for an elliptical cone C with major and minor semiangles β_1 and β_2 to lie in the interior of some octant. For circular cones $\beta_1=\beta_2=\beta$, this condition reads $2\tan^2\beta<1$ i.e., $\cos\beta>\sqrt{2/3}$, i.e. angular diameter 2β of C_B is less than $2\cos^{-1}\sqrt{2/3}$; therefore (2.3.3) shows that *the bound given in* (2.2.7) *is the best possible.* Note also that any nondegenerate bilinear form $B: V^2\times V^2\to W^3$, $\langle\,\rangle$ having a circular cone C_B with an angular diameter $\geq 2\cos^{-1}\sqrt{2/3}$ but less than $\pi/2$ gives a counterexample to the "generalized Cauchy-Schwarz inequality" formulated in (2.2.5). We remark that (2.3.3) extends to all $B: V^2\times V^2\to W^3$, $\langle\,\rangle$ with $0\notin C_B$ if we define $\beta_1=\alpha/2$, $\beta_2=0$ (resp. $\beta_1=\beta_2=\pi/2$) whenever C_B is a planar sector of angle $0\leq\alpha<\pi$ (resp. a 2-dimensional subspace with origin deleted); by (2.1.3) we know that these are the only possible degenerate cases. It seems likely that one has a characterisation analogous to (2.3.3) of all nondegenerate symmetric bilinear forms $B: V^n\times V^n\to W^{n(n+1)/2}$, $\langle\,\rangle$ with R_B positive definite.

(2.3.4) *$B: V^n\times V^n\to W^p$, $\langle\,\rangle$ is positively curved iff the cone C_β of each restriction $\beta: \nu^2\times\nu^2\to W^p$, $\langle\,\rangle$ of B to a 2-dimensional subspace $\nu\subseteq V$, can be contained in the interior of some octant* (*i.e. some* 3*-orthant*) *of W^p, $\langle\,\rangle$.* Arguing as in (2.2.7) we see that B is positively curved iff each restriction $\beta: \nu\times\nu\to\omega$, $\langle\,\rangle$ is positively curved. If $\dim\omega\leq 2$, then it is clear that C_β lies in the interior of an octant of W^p, $\langle\,\rangle$ iff angular diameter of C_β is less than $\pi/2$. If $\dim\omega=3$ then C_β can lie in the interior of only those octants of W^p, $\langle\,\rangle$ which are octants of ω^3, $\langle\,\rangle$. The result now follows by using (2.2.5) and (2.3.3).

3. An example. Let v_1, v_2, v_3, v_4 (resp. w_1, w_2, w_3) be a basis (resp. orthonormal basis) of V^4 (resp. W^3, $\langle\,\rangle$); further let us equip Λ^2V with the basis $\omega_1=v_1\wedge v_2$, $\omega_2=v_1\wedge v_3$, $\omega_3=v_1\wedge v_4$; $\omega_1^*=v_3\wedge v_4$, $\omega_2^*=v_4\wedge v_2$, $\omega_3^*=v_2\wedge v_3$. Let $*: \Lambda^2V\times\Lambda^2V\to\mathbf{R}$ denote the bilinear form whose matrix with respect to this basis is

$$\begin{bmatrix} 0_{3,3} & I_{3,3} \\ I_{3,3} & 0_{3,3} \end{bmatrix}; \tag{7}$$

it is well known that $*(\omega, \omega) = 0$ whenever ω is decomposable. For each $\lambda \in \mathbf{R}$, $B_\lambda : V^4 \times V^4 \to W^3$, $\langle\ \rangle$ shall denote the symmetric bilinear form for which $B_\lambda(v_1) = B_\lambda(v_4) = 2w_2$, $B_\lambda(v_2) = 2w_2 + \lambda w_3$, $B_\lambda(v_3) = 2w_2 + w_3$, $B_\lambda(v_1, v_3) = B_\lambda(v_2, v_4) = w_1$ and $B_\lambda(v_i, v_j) = 0$ for all other pairs $\{v_i, v_j\}$. Using (1) one easily computes the matrix of R_{B_λ} (with respect to the basis of $\Lambda^2 V$ chosen above) to be

$$\begin{bmatrix} \operatorname{diag}(4,3,4) & \operatorname{diag}(1,0,-1) \\ \operatorname{diag}(1,0,-1) & \operatorname{diag}(4,3,4+\lambda) \end{bmatrix}. \tag{8}$$

B_λ is positively curved iff $\lambda > -4$; R_{B_λ} is positive definite iff $\lambda > -3.75$. The matrix of $R_{B_\lambda} + *$ (i.e. the sum of (7) and (8)) can be diagonalised by a couple of elementary row and column operations to get $\operatorname{diag}(4,3,4,3,8/3,4+\lambda)$; thus for any nonzero decomposable ω, $R_{B_\lambda}(\omega, \omega) = (R_{B_\lambda} + *)(\omega, \omega)$ is positive if $4 + \lambda > 0$. Conversely, $\langle B_\lambda(v_2), B_\lambda(v_3) \rangle = 4 + \lambda$ shows that the angle between $B_\lambda(v_2)$ and $B_\lambda(v_3)$ is less than $\pi/2$ only if $4 + \lambda > 0$. The second part follows by noting that the matrix (8) can be reduced to $\operatorname{diag}(4,3,4,15/4,3,3.75+\lambda)$.

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