Plain Geometry & Relativity

K S Sarkaria

October 26, 2013. *If we are constrained to a bounded region* Ω *of euclidean space* (of dimension $n \geq 2$) *the parallel postulate is no longer true* : a point P not in a codimension-one flat L is plainly in infinitely many flats L_i which do not meet L inside Ω. So, ***non-euclidean geometries are dime-a-dozen,*** for instance – see Figure 1 – that sheet of paper on which we ask schoolchildren to do all those constructions from Euclid has such a geometry; likewise, *any open* n-*ball* B^n *of finite radius* c, and it is this ball geometry that will concern us from here on.

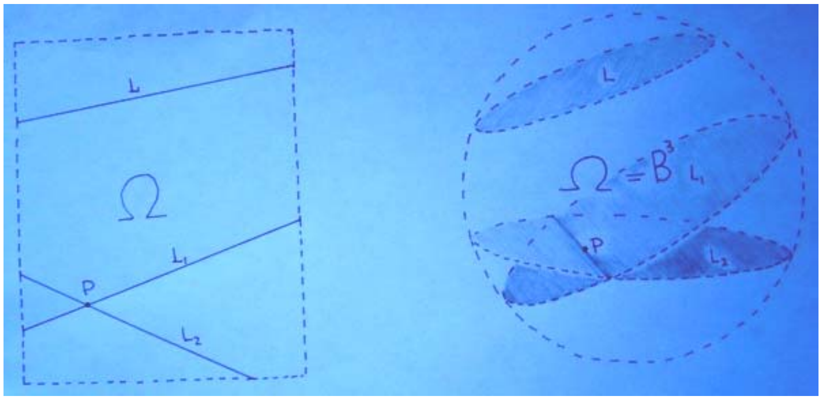

Figure 1

In the euclidean case c = ∞, of all the affine **reflections** in a flat, we pick the one in the direction orthogonal to the flat, and call two subsets of n-space congruent iff they are related by a finite sequence of such reflections. The only affine reflections of n-space which preserve a ball B^n of radius c < ∞ are the ones in, and orthogonal to, the flats through its center b. However a general and canonical definition emerges if we recall that, to **linearize** the affine transformations of n-space, we should consider it as a flat in *a vector space of one dimension more*—see Figure 2—whose 0 is outside this flat, and identify each point with the *ray* from 0 through that point.

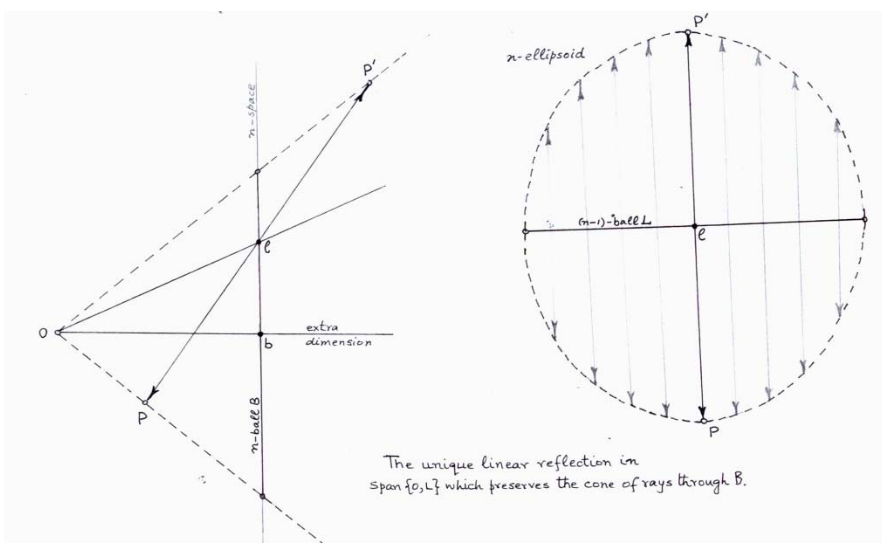

Figure 2

In each flat L ***of the ball*** B ***there is a unique linear reflection which preserves the cone of rays through*** B**.** Using the extra dimension, we have now a one-parameter family of lines orthogonal to L, and passing through its centre ℓ. One of these orthogonal lines has ℓ as the mid-point of its intersection PP' with the cone (note that it is the line such that {0,P,2ℓ, P'} is a parallelogram, which gives an easy construction). The linear map which is the identity on L and which takes P to P' reflects each point of the vector space in span{0,L} in the direction parallel to the segment PP'. Furthermore it restricts, on the flat containing L and PP', to an orthogonal reflection preserving its ellipsoidal intersection—see Figure 2—with the cone of rays through B. So it preserves this cone, and no other linear reflection in L does the same because, the composition of two distinct linear reflections in L restricts to a nonzero translation in any flat parallel to span{0,L}, but our cone does not contain a complete line, q.e.d. So, ***there is a unique linear reflection of the cone which switches the ray through the centre*** b ***with any other ray***, the corresponding flat L of the ball having its centre ℓ suitably between these two rays on the 2-plane determined by them. Also, ***this homogenous but non-euclidean geometry 'explains' euclidean geometry*** : for these reflections approach orthogonal reflections of n-space when c tends to infinity. To wit, the geometry in which subsets of a ball of finite radius are **congruent** iff the sets of rays passing through them are related by a finite sequence of linear reflections preserving the cone.

In physics, ***the rays identify with galilean observers***, to the observer S who considers the ray through the point b of the flat as representing his state of rest, and uses as *time* t *the linear function* which is 1 on this flat, each ray S' represents all particles moving in a fixed direction at the same speed v. Relativity is the dictum that, ***any other observer*** S' ***observes the mirror image of what*** S ***observes***, under the linear reflection switching these rays : *so the time* t' *and the euclidean space* t' = 1 *of* S' ≠ S *must be the transforms of* t *and* t = 1 *under this reflection.* Pragmatic considerations of speed measurement tell us that we should only admit **a ball's worth of observers**, which implies that neither the time nor the space of S' ≠ S are the same as that of S, for example, from Figure 2 it is clear that the mirror image b' of b, which must be in t' = 1, is not in t = 1 : the ***"notions of absolute space and absolute time have no empirical definitions"***, they are only left-overs from the limiting and unrealistic case c = ∞.

The observer S *identifies the points of* span{S,S'} *with their* ***cartesian coordinates*** (t,x) *with respect to* $\overrightarrow{0b}$ *and the unit vector* $\overrightarrow{be}$ *in* t = 1 *towards* S' : so b = (1,0), e = (1,1), ℓ = (1,u) for some u, and S' is the ray through (1,v). The reflection keeps ℓ and all vectors parallel to L fixed. Lemma: *if the sides of a parallelogram have slopes* ±c *then the product of the slopes of its diagonals is* c^2. So PP' has slope c^2/u and b' is on the line through b with this slope such that the mid-point of bb' satisfies x = ut, which gives b' = $((c^2 + u^2)/(c^2 – u^2), 2uc^2/(c^2 – u^2))$. The ray S' passes through b', so v = $2uc^2/c^2+u^2$, and this quadratic in u can be solved, using u < c, to write u in terms of v. Let $\gamma = (c^2 + u^2)/(c^2 – u^2)$, then b' = (γ, γv) and $1/\gamma^2 = 1 – v^2/c^2$. So $\gamma t – (\gamma v/c^2)x = 1$ on ℓ and b', also this linear function is zero on vectors parallel to L, so it is the transform of t : *the observer* S' *has time* $t' = \gamma t – (\gamma v/c^2)x$. His space t' = 1 is the flat spanned by L and b' with the same metric on L, but $\overrightarrow{b'e'}$ is a normal *unit* vector : *to* S'*, the "ellipsoidal"* (see Figure 2, but this is a different section of the cone) *image* B' *of* B *is the ball in* t' = 1 *with centre* b' *and radius* c. The observer S' identifies the points of span{S',S} by their coordinates (t',x') with respect to $\overrightarrow{0b'}$ and the unit vector $\overrightarrow{b'e'}$ in t' = 1 towards S. *One has* x' = γvt – γx because $\overrightarrow{be}$ and its image $\overrightarrow{b'e'}$ = u^{-1}(ℓ – b') have (t,x) coordinates (0,1) and $(– \gamma v/c^2, – \gamma)$ respectively. Since the unit vector $\overrightarrow{0b'}$ of S' runs from the t = 0 to the t = γ line, while his unit vector $\overrightarrow{b'e'}$ runs from the x = 0 to the x = – γ line, and γ > 1, S ***deems the clocks of*** S' ***to be slower, and his rulers in the*** x-***direction to be contracted***—the rulers in directions parallel to L are unaffected—***by the factor*** γ, and S' observes the same about the unit vectors $\overrightarrow{0b}$ and $\overrightarrow{be}$ of S in his (t',x') coordinates because, this being a reflection, we also have $t = \gamma t' – (\gamma v/c^2)x'$ and x = γvt' – γx'.

However, the proper time τ ***and space*** τ = 1 ***that we should use are absolute!*** *In analogy with* c = ∞ *this space should consist of all the mirror images* b' *of* b*, with* τ *linear on all rays.* So, ***for*** c < ∞***, proper time is not linear, but now it dictates a distance on the ball which is preserved by all the linear reflections of its cone!*** The (t,x) coordinates of the mirror images b'(v) in any plane through ray S are (γ(v), γ(v)v), but $\gamma(v)^2(1 – v^2/c^2) = 1$, so these points form the hyperbola $t^2 – x^2/c^2 = 1$, t > 0. Therefore, if we decompose each vector parallel to the time t and the euclidean space t = 1 of S, τ^2 *extends to the quadratic form* $(t; \mathbf{x}) \to t^2 – \mathbf{x}.\mathbf{x}/c^2$. Since PP' has slope c^2/u, *our linear reflection is orthogonal to its mirror with respect to the associated bilinear form* $(t_1; \mathbf{x_1}) \times (t_2; \mathbf{x_2}) = t_1t_2 – c^{–2}\mathbf{x_1}.\mathbf{x_2}$. So, it preserves this form. Also, it replaces t by the time t′ of S′ and t = 1 by the euclidean space t′ = 1 of S′. So *this form is the same as the bilinear form* $(t'_1; \mathbf{x'_1})$ x $(t'_2; \mathbf{x'_2}) = t'_1t'_2 - c^{-2}\mathbf{x'_1}.\mathbf{x'_2}$ *of* S′. To S′ too, τ = 1 is obtained by revolving $t'^2 - x'^2/c^2 = 1$ around his time axis. On vectors parallel to lines which cut the cone's boundary twice the invariant quadratic form τ^2 is negative. *The positive square root of* $-\ c^2\tau^2(\overrightarrow{b'b''})$ *gives us an invariant distance between points* S′ *and* S″ *of the ball.* The coefficient c^2 ensures that, ***when the radius tends to infinity this non-euclidean distance approaches the euclidean distance*** : to see this it suffices to check that $-\ c^2\tau^2(\overrightarrow{bb'})$ approaches v^2. So this distance gives us the **proper speed** -- i.e. the proper length for the difference $\overrightarrow{b'b''}$ of their 'absolute velocities' $\overrightarrow{0b''}$ and $\overrightarrow{0b'}$ -- separating the observers S″ and S′.

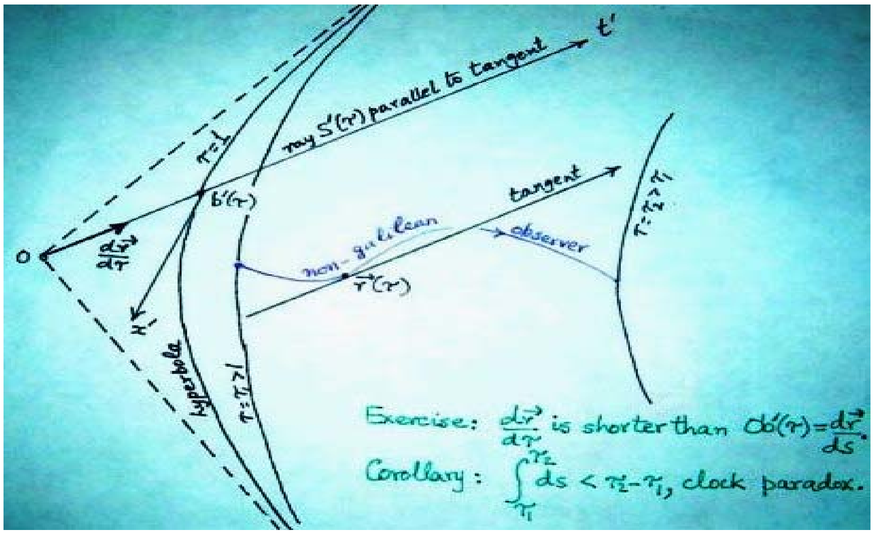

Figure 3

A particle's à priori or ***cartesian absolute motion*** may be any smooth directed arc C with τ strictly increasing, i.e., a smooth vector function $\mathbf{r}(\tau)$, with each $d\mathbf{r}/d\tau$ nonzero along a ray S′(τ). Its **absolute velocity** $d\mathbf{r}/ds$ at proper time τ, i.e., the 'absolute velocity' $\overrightarrow{0b'}$ of S′(τ), is longer – Figure 3 – than $d\mathbf{r}/d\tau$ if 0 is not on tangent, so ds < dτ. Hence the clock paradox: *the **elapsed time** ∫ds on a clock carried by a non-galilean observer is less than his proper life-time!* Further, the newtonian "the rate of change of momentum is equal to the force" suggests $d/ds(m\ d\mathbf{r}/ds) = \mathbf{F}(s)$, where the constant m is called the **proper mass** of the particle, and the right hand side, the **absolute force** on the particle.

An observer S will write $\mathbf{r} = (t; \mathbf{x})$ to split the last equation into temporal and spatial components : $d/dt(m\ dt/ds) = T\ ds/dt$ and $d/dt(m\ dt/ds\ \mathbf{v}(t)) = ds/dt\ \mathbf{X}$, using $d\mathbf{x}/dt = \mathbf{v}(t)$ and $\mathbf{F} = (T; \mathbf{X})$. Taking the spatial equation as Newton's law, ***the observer*** S ***deems*** $ds/dt\ \mathbf{X}$ ***as the force on the particle, and*** $m\ dt/ds$ ***as its varying mass*** m(t)! Since $d\mathbf{r}/dt = (1; \mathbf{v}(t))$ is the vector from 0 to (1, v(t)) in the plane of S and S′(τ), and γ(v(t)) times this vector is $\overrightarrow{0b'} = d\mathbf{r}/ds$, ***we have*** $dt/ds = \gamma(v(t))$ ***on the arc*** C. Also differentiating $d\mathbf{r}/ds$ x $d\mathbf{r}/ds \equiv 1$ we see that, ***the absolute acceleration is always orthogonal to the absolute velocity***, i.e., $d^2\mathbf{r}/ds^2$ x $d\mathbf{r}/ds = 0$, i.e., $d^2t/ds^2\ dt/ds - (d^2\mathbf{x}/ds^2.d\mathbf{x}/ds)/c^2 = 0$ in the components of S. Multiplying by mc^2 we obtain $c^2T = \mathbf{X}.\mathbf{v}$, therefore $T\ ds/dt\ c^2 = ds/dt\ \mathbf{X}.\mathbf{v}$ = the rate of working of the force on the particle = the rate of change of its energy. So the temporal equation of motion shows S that, ***the energy of the particle is*** $E = m\ dt/ds\ c^2 = m(t)c^2 = m(1 - v(t)^2/c^2)^{-1/2}c^2 = mc^2 + ½\ mv(t)^2 + \ldots$!

But, $m(t)\mathbf{v}(t)$ plus a nonzero vector **a** also satisfies the spatial equation, why not this rest-momentum? Because it violates mirror-relativity, the opposite direction is just as good. And, $m(t)c^2$ plus a nonzero scalar k satisfies the temporal equation, but energy is additive in mass, this would add k/c^2 to m(t), which we have accepted. Excepting some other similar points, that one can debate endlessly, we have now covered all the basics, and our treatment was *plain* indeed : all our key arguments used only *plane* geometry, though the results are quite general!

Notes

1. The non-euclidean geometry of a polytopal and possibly non-convex region Ω of euclidean n-space is also worth pursuing, especially because of its uncanny ***galois symmetries***, cf. Sullivan, M.I.T. Notes (1970).

2. The word group makes us forget that the basic symmetry is that of Euclid's *pons asinorum*, viz., a reflection, for example for Galois it was a transposition of roots, and we know now that practically all non-abelian **finite simple groups** can be realized by using the linear reflections of the cone of a 2-ball, cf. *How I learnt some well-known folklore* (2010). Besides, though a reflection takes us abruptly into the oppositely oriented mirror world, it is ***magic*** that an even number of such hops will do the same job as any continuous motion which preserves our geometry!

3. Linearization by adding extra dimension(s) is like adding 0 to the positive numbers ... we all use it ... I used it (1992, 1997, 2000) to look at some problems of convex geometry with only partial success ... maybe because I'd turned away from number fields which had given me the initial idea? This thought comes because number fields give many examples of **discrete subgroups Γ of linear transformations of the cone** with B^n/Γ a closed n-manifold, and the much finer use—which I've still not quite understood—of galois symmetry in Deligne and Sullivan, *Fibrés vectoriel complexes à groupes structurel discret,* C. R. Acad. Sci. Paris 281 (1975) 1081-1083, shows further that there are hordes of these groups for which this closed manifold is almost parallelizable.

4. Appollonius knew from the sun-dial in his garden in Perga (Turkey) that ***the shadow of a sphere is an ellipse***, the point F common to the sphere and the table being a focus; the other focus F' is the point of PP' – see Figure 2 – on the *other* circle in the cone which touches the sides of 0PP'; but in his treatise he used a 2-dimensional but less natural definition of an ellipse which unfortunately is the only one that is taught in schools today.

5. My scientific journey started with Henry Thomas and Dana Lee Thomas, *Living biographies of great scientists* (1959), my First Prize in Aggregate during my first year, 1960-61, in Government College, Chandigarh. Its last biography led me to mail order a popular text-book on relativity from Bombay, from which I was sharp enough to deduce that the prize-winning "mathematics" I knew was not mathematics at all, which led me in turn to mail order, from Varanasi this time, that classic of Goursat's which I used again in *Straight to Mecca*. I should mention also that my mirror-formulation of relativity is ***slightly stronger***, there is no preferred orientation on spacetime.

6. Again, my use of the phrase proper time is not standard, often it means elapsed time along C. That τ defines a proper speed between galilean observers is nice, but ***this distance only satisfies the triangle inequality infinitesimally*:** at each point b' of the smooth submanifold τ = 1 it gives us the euclidean metric of its tangent hyperplane t' = 1. So by integrating this infinitesimal we can assign a length to any smooth curve of this ***riemannian manifold*** between two given points, and the infimum of these lengths gives us another invariant distance satisfying this inequality.

7. The haloed, but nevertheless arbitrary, newtonian tradition of considering only motions given by second order differential equations seems even more arbitrary now : ***for*** $c < \infty$ ***we have a nice and complete classification of generic cartesian absolute motions parametrized by elapsed time!*** This is suggested by the final paragraphs, where we stuck to this tradition, but what gave $E = mc^2$ was the cartesian orthogonality $d^2\mathbf{r}/ds^2 \times d\mathbf{r}/ds = 0$ with respect to our non-degenerate bilinear form. There seems no reason to stop at two, we can keep on normalizing and differentiating à la Frenet and Serret till we have a full complement of n+1 orthonormal vectors all along our generic C, and then using this frame define the n curvatures which will characterize C up to a composition of orthogonal reflections in flats of spacetime. A quick recap of the classical classification of space curves is in my old class notes on *Differential geometry* (1982?), and in the book by Klingenberg, *A course in differential geometry* (1978), it is shown that everything works just as well for any positive definite quadratic form over the reals.

(*contd.*)