Oxymorons No More!

K S Sarkaria

July 23, 2013. To recap, a geographical problem led us to define *bend-distance*, which not only clarified Gleason's delicate proof of the *automatic continuity of a non-negative frame function on* S^2, it had also shown us that, implicit in his proof, there was an infinity of configurations which can be used to make finite startling subsets. The two smallest configurations, viz., the ones with 2 and 3 bends, for the case of maximum possible angle between N and W, that is, for $\sin^{-1}(1/3)$ and $\tan^{-1}(1/2)$ respectively, are shown in Figure 1 : both as centrally projected points in the tangent plane at the north pole, as well as the corresponding ***points on a cube*** circumscribing the unit 2-sphere.

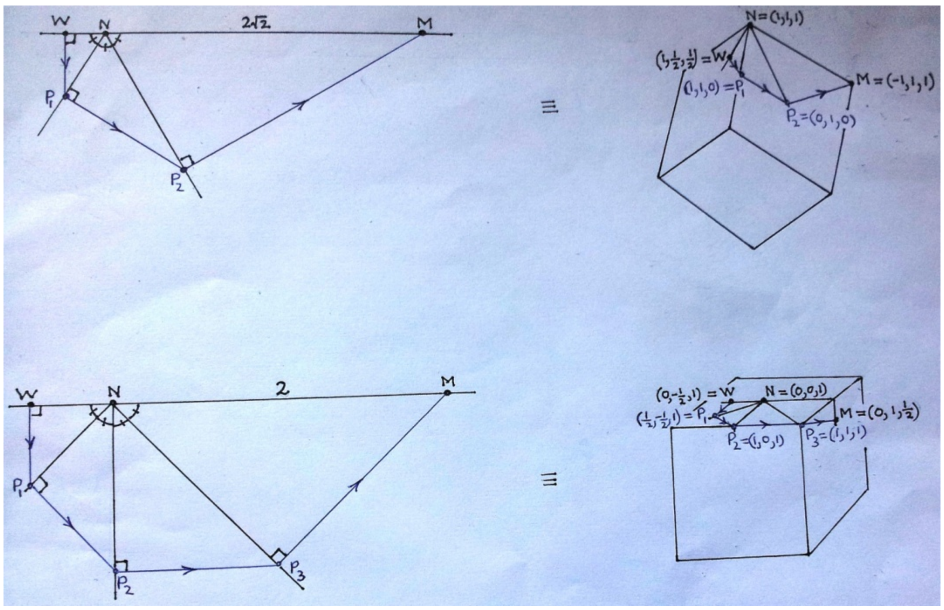

Figure 1

That it is the same set of *rays* from the centre which is depicted on the cube is clear for the second configuration, even the *points* N, W, P_1, P_2 and P_3 are the same, only the ray to M cuts the cube in the adjacent facet. For the first configuration, calculations show that the angular distances between corresponding pairs of rays coincide for the two depictions. We turn next to Figure 2 below which displays *the smallest cubically symmetric set containing the second configuration* – note that it has 37 antipodal pairs of points or lines or pure states, and of these, 25 are generated by the first configuration – together with a quick proof that, ***this set of* 37 *lines is almost startling*:-**

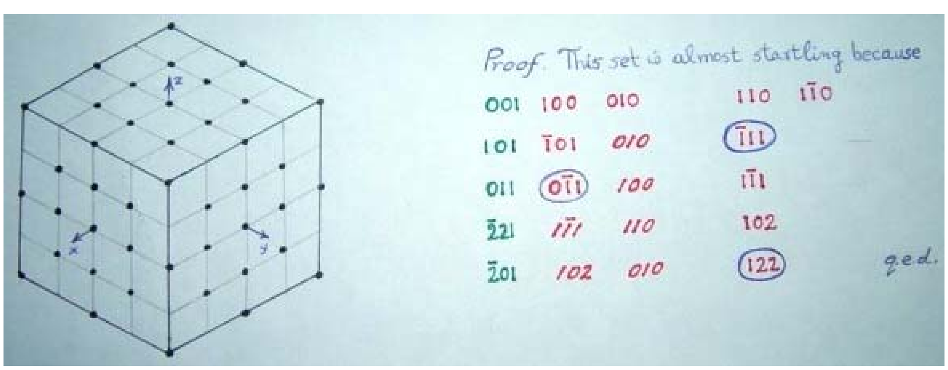

Figure 2

The notation $\bar{2}01$ = (-½ , 0, 1), etc., is used in the above table. Assume if possible a colouring of the rays through these 74 points as *green* (true) and *red* (false) such that rays orthogonal to any green ray are red, and there is exactly one green in any orthogonal triad of rays. We start with 001, 100 and 010 which form such a triad; by symmetry we may assume that 001 is green; so all other rays orthogonal to 001, e.g. 110 and $1\bar{1}0$, must be red. We look next at the triad 101, $\bar{1}01$ and *010* (italics means we know it is red); by symmetry we may assume it is 101 which is green while $\bar{1}01$ is red; so all other rays orthogonal to 101, e.g. $\bar{1}11$, must also be red. Symmetry is used likewise in the third row of the table. After that, it is only the known redness of the two italicized rays that is used in each row. A contradiction appears after the fifth green : the circled orthogonal triad of rays are all in red.

The above set is *not* startling: there is no obstruction to the above procedure for colouring its rays green and red in such a way that there is exactly one green ray in each orthogonal triad. For example, in the fourth row, the greenness of $\bar{2}21$ now no longer implies that 102 should be red, because the line perpendicular to these two orthogonal lines is not in our set of rays. However, by roping in these lines too, equivalently, *by marking* 8 *more points in each facet of our cube, viz., those with the other two coordinates* ± 1/5 *and* ± 2/5, we do get a cubically symmetric ***bigger set of* 61 *lines which is startling***, with the exact same table serving as our proof of this fact.

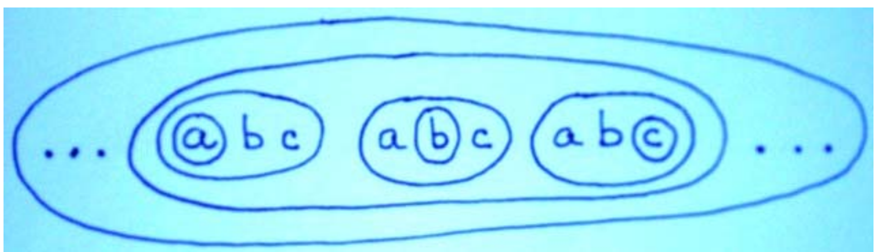

Figure 3

What you see above is—in G Spencer-Brown's notation, which we met in *Tweaking*—a statement which is true if and only if, for every triad {a, b, c} used in it, exactly one of these three variables is true. So, by assigning a variable to each of our 61 lines, and using all 46 (check!) orthogonal triads of these lines, Figure 3 *associates to our startling set an* ***oxymoron****, i.e., a logical statement which is always false, irrespective of the truth or falsity of its* 61 *variables* (and so, if we negate it by erasing its outer string, we obtain a tautology in 61 variables, i.e., a logical statement which is always true; on the other hand, if we similarly use the 37 lines and their 22 orthogonal triads, Figure 3 is not an oxymoron, it is false if and only if we take care not to assign the truth value 1 to any two orthogonal variables). However, if 'negation' means orthogonal complement of a subspace, and 'disjunction' sum of subspaces (which are partial sums of the same orthogonal triad of lines), then Figure 3 evaluates, on the startling set of 61 lines, to all of 3-space. Therefore, *this oxymoron of ordinary logic is* ***not*** *an oxymoron of startling or quantum logic ...*

Which shows that, equipped with these operations, even {subspaces of 3-dimensional space} do not embed in a boolean algebra, a stark reaffirmation of von Neumann's result that, ***it is impossible to introduce hidden variables*** *into the infinite dimensional linear description of the subatomic world.* Besides its physical successes, this description has inspired many beautiful discoveries in all of mathematics. Nevertheless, there is a pervasive feeling that something is missing. Feynman put it thus, *"I think it is safe to say that no one understands quantum mechanics,"* or else again, *"If someone tells you they understand quantum mechanics, then all you've learnt is that you've met a liar."* And J S Bell, who gave a number of such proofs, hoped that, ***"what is proved by the impossibility proofs is lack of imagination."*** In particular, he saw nothing wrong in the direction in which its discoverer, Louis de Broglie, had launched wave mechanics in his thesis. The now mainstream formulations of Schrödinger and Heisenberg, which von Neumann put on a firmer footing using hilbert spaces, came later. But, by the time Feynman had developed path integrals in his thesis, manifold theory was developed enough to go back in time. Indeed, cartesian strings reappeared, and shed new light on manifold theory, but—alas!—as far as physics is concerned, cartesian clarity was *not* the aim, a hunt was on for much bigger game, than the mere explaining of known facts in an elegant way ... a way more in tune with the intuitions that had led to these discoveries in the first place ... this rewriting is not easy, but it needs to be done, it can be done, so eventually it will be done, as ... *"we beat on, boats against the current, borne back ceaselessly into the past."*

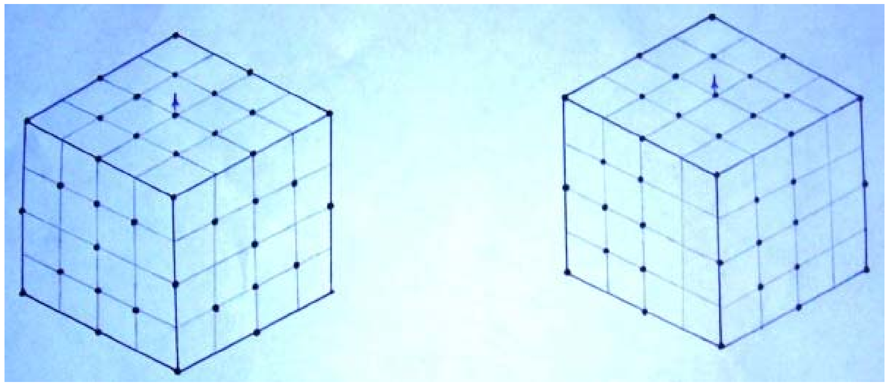

Figure 4

Returning to ***Gleason's cube***, as I like to call Figure 2 because of its affinity with his 1957 paper, its five-line tabular proof is shorter than the nine-line 1990 proof which had ushered in ***Peres' cube*** – an almost startling set that we met before in *Startling Logic* – which is just as symmetric, but has only 33 lines, however some lines are irrational. To get a rational example with less than 37 lines, as seems necessary, we'll give up on full cubic symmetry. *The two subsets of Gleason's* 37 *lines shown in* Figure 4 *are both almost startling*: the one on the left, the ***Bub-Schütte cube***, has 33 lines and is preserved by only 16 of the 48 symmetries of the cube; the other, ***Conway's cube***, has two less lines, and is preserved by only half of these 16 symmetries. In his 1996 paper Bub writes that the proof of the first was essentially all there in a 1965 letter of Schütte to Specker, but for some reason the subsequent 1967 paper of Kochen and Specker did not mention this and used a much bigger example with 117 lines. Conway found his 31 lines in 1990. I've not made tabular proofs for these two cubes, because they are longer, and such work is best left to a computer. And indeed, lots of tech-savvy people are engaged in such work. In particular, *it seems the examples of* Figure 4 – both of them! – *are the current world-record holders*: Conway's 31 lines is the smallest known 3-dimensional almost startling set, while the startling set of 49 lines generated by the Bub-Schütte cube is the smallest startling set known. This continuing interest in small startling sets has been stimulated even further recently by ***"The strong free will theorem"*** of Conway and Kochen. A contemporary result which shows yet again how very ceaselessly our boats are being borne into the distant past, for there is an unmistakable cartesian ring to it : it says roughly that, *if we humans have free will, then elementary particles also have free will* ! For a precise statement and a proof see the Notices of the American Mathematical Society, 56 (2010), pp. 226-232. This proof uses the existence of a finite almost startling set. To make it self-contained, a colourful proof that the Peres' cube is almost startling is included in this paper. However, in my opinion, the original tabular proof of Peres was simpler, and simpler still would have been to use Gleason's cube instead.

www.kssarkaria.org

Notes

1. F Scott Fitzgerald's *beating on into the past*—wisdom often wears an oxymoronic garb—appeals to me; as does the idea that human thinking consists of feed-ins to and feed-backs from the past, temporal strings if you will; as does G Spencer-Brown's observation that, mathematics is self-analysis, *discovering what we know*!

2. What I knew about Schrödinger and Heisenberg in 1976-77, when I tried to write wave mechanics as manifold theory—only some of this work was typed up, but the initial pages appeared duly in print (26 years later!) as *Stationary States* (2002)—is now mostly in the recesses of my memory; but, true to what I wrote in *The Joys of Forgetting,* I've been trying to dig it up again, and if possible polish it into something worthwhile ...

3. Since they represent rays through those dots, we should perhaps equip our cubes with the *angular distance* between the corresponding points on the unit 2-sphere. *The cube's surface metric is different* : for example, there are *two* shortest paths from the pole (0,0,1) to the point (1,1,0) on its square-shaped equator, or from (1,1,1) to (½, ½, -1) on its hexagonal—cf. *"Eyes"* on p. 7 of *"213, 16A" and Mathematics* (2010)—equator, and even between antipodal points there are only *finitely* many such paths. However, rectilinear angles in facets equal in magnitude the corresponding curvilinear angles on the 2-sphere : so a *cubical globe*, more generally a polytopal globe, may not be a bad idea for cartographers to look into, *it can be easily assembled from an atlas of conformally accurate maps*!

4. Consider the two integral points ± (a, b, c) nearest to the origin on a rational line. The line passes through a rational point of S^2 iff $a^2 + b^2 + c^2$ is a perfect square. Going mod 4 shows that this happens only if *exactly one* of these three integers is odd. So, *the rational unit* 2-*sphere is not startling* : colour an antipodal pair green if c is odd, otherwise red ! This pretty proof is from D A Meyer, *Finite precision measurement nullifies the Kochen-Specker theorem,* Physical Review Letters 83 (1999) pp. 3751-3754, which has more on dense but non-startling subsets of S^2.

5. Turning to J Bub, *Schütte's tautology and the Kochen-Specker theorem,* Found. Phy. 26 (1996) pp. 787-806, its pp. 788-789 misinform us that Kochen-Specker (1967) had claimed that Figure 3 is an oxymoron for their set of 117 lines and its 43 triads : actually, see pp. 86, 69 of their paper, they use the generated startling set D. However, it is almost as easy to write, *for any almost startling set*—see Manin, p. 92—*a tautology in as many variables, by using, besides all orthogonal triads, also all orthogonal pairs.* Again, though Schütte's formidable tautology is re-written by Bub in painful detail using disjunctions and negations—so can easily be written à la G Spencer-Brown—the "It now follows easily (but tediously)" on page 793 did not make me any *wiser*—see Manin, p. 51—about it. Anyway, this tautology says that it is not possible to assign to some lines and planes determined by {100, 010, 011, 101, 110, $\bar{1}11$, $1\bar{1}1$, $11\bar{1}$, 111, 201, 102} the colours green and red in such a way that the line 100 is green, *etc., etc., etc.* Which suffices to show that there is no colouring of the 33 lines of the Bub-Schütte cube such that 100 is green, any line orthogonal to a green is red, and in any orthogonal triad of lines there is exactly one green. That this cube is almost startling then also follows from the table given in Figure 2, which rules out the possibility 001 green, leaving only the symmetrical possibilities 100 green or 010 green. So, Figure 3 gives us an oxymoron even if we use the generated startling set of 49 lines—a subset of our 61 lines—and its 36 triads, and these lines and triads are listed in full in Bub's paper.

6. M Pavičić et al., *Kochen-Specker vectors,* J. Phys. A : Math. Gen. 38 (2005) pp. 1577-1592, tells us that the startling set D of Kochen and Specker had 192 lines, and those generated by the cubes of Bub-Schütte, Conway and Peres have 49, 51 and 57 lines; and papers by this and other groups, J Oukanine et al., A Cabello et al., ... , inform us that no almost startling example smaller than Conway's is known. I'll conclude by saying however—most irreverently in this day and age!—that now and then even those who have 'never done five lines of any calculation without making a mistake' can beat computers : see, for example, *A "nice" map colour theorem* (2002).