8. *The three things that we left to the reader in the text are also easy to check.* The Lemma holds because the product of the slopes of the diagonals of the parallelogram $\{(0,0),(t,-ct),(t',ct'),(t+t',-ct+ct')\}$ is $\frac{ct'+ct}{t'-t} \times \frac{-ct+ct'}{t+t'} = c^2$. Again, $-c^2\tau^2(\overrightarrow{bb'}) = -c^2\tau^2(\gamma-1,\gamma v) = -c^2((\gamma-1)^2 - \gamma^2 v^2/c^2) = -c^2((\gamma-1)^2 + \gamma^2(1/\gamma^2-1)) = -c^2(2-2\gamma) = -2c^2(1-(1-v^2/c^2)^{-1/2}) = -2c^2(-v^2/2c^2+\cdots)$ approaches v^2 as $c \to \infty$. And, for the Exercise in Figure 3 note that $d\overrightarrow{r}/d\tau$ along C at the point of tangency $(t',a), a \neq 0$ – using coordinates (t',x') in the plane containing the tangent line and the parallel S' – equals the same quantity along the tangent line at $\overrightarrow{r} = (t',a)$. So it is equal to the vector $(1,0) = \overrightarrow{0b'}$ times $dt'/d\tau$ along the tangent line at this point, and from $\tau^2 = t'^2 - a^2/c^2$ we see that this derivative is equal to $\tau/t' = (1-a^2/(c^2t'))^{1/2} < 1$. Also, *our clock paradox implies the one usually stated,* because, if the cartesian motion C begins and ends on any ray S, then $\tau_2 - \tau_1 = t_2 - t_1$.

9. *Velocity addition formula.* Given a cartesian motion C, the time t of any observer S increases strictly on it, so it has equation $\mathbf{r}(t) = (t;\mathbf{x}(t))$ and $d\mathbf{x}/dt$ is its varying velocity as observed by S. For example S can use an orthogonal basis of his euclidean space $t=1$ with respect to which $\mathbf{x}(t)$ has cartesian coordinates $(x_1(t),x_2(t),\ldots,x_n(t))$ and measure the n components $(dx_1/dt,\ldots dx_n/dt)$ of $d\mathbf{x}/dt$. Likewise, the observed velocity $d\mathbf{x}'/dt'$ of C as measured by another *framed observer* S', that is one equipped, besides his unit time vector $\overrightarrow{0b'}$, with an orthogonal basis for his euclidean space $t'=1$, gives us another n-tuple $(dx'_1/dt',\ldots dx'_n/dt')$. The two bases of the $(n+1)$-dimensional vector space used by S and S' are related by a *matrix* $A = [a_{ij}], 0 \leq i,j, \leq n$, i.e.,

$$\begin{aligned} t &= a_{00}t' + a_{01}x'_1 + \cdots + a_{0n}x'_n,\\ x_1 &= a_{10}t' + a_{11}x'_1 + \cdots + a_{1n}x'_n,\\ &\cdots\cdots\cdots\\ x_n &= a_{n0}t' + a_{n1}x'_1 + \cdots + a_{nn}x'_n; \end{aligned}$$

from which we get, for all $1 \leq i \leq n$,

$$\frac{dx_i}{dt} = \frac{a_{i0} + a_{i1}dx'_1/dt' + \cdots + a_{in}dx'_n/dt'}{a_{00} + a_{01}dx'_1/dt' + \cdots + a_{0n}dx'_n/dt'},$$

a formula relating the velocity components of the same motion C as measured by two framed observers. *Often framed observers are called observers, but for us an observer*—that is a ray per the fourth paragraph of the text—*has an* $O(n)$ *worth of frames.* Components of an observed velocity depend on the frame, for example, if we reverse a basis vector, that component changes its sign. Given any other observer S', the central observer S has an $O(n-1)$ worth of frames in which the observed velocity of S' is $(v,0,\ldots,0)$ with v positive. If S uses one of these, and S' the reflected frame as in the fifth paragraph of the text, then $t = \gamma(v)(t' - v/c^2x'_1), x_1 = \gamma(v)(vt' - x'_1), x_2 = x'_2, \ldots, x_n = x'_n$. But S' can, if he wants, 'correct' his reversed orientation by reversing his x'_1-axis, when A is given instead by $t = \gamma(v)(t' + v/c^2 x_1')$, $x_1 = \gamma(v)(vt' + x_1')$, $x_2 = x_2', \ldots, x_n = x_n'$ (and it is usually this case only of the above formula, with C galilean, which is called the velocity addition formula), and more generally, S' can transform the reflected frame by any orthogonal transformation he likes of $t' = 1$.

10. Though in the proof in the third paragraph of the text we temporarily assumed the central ray orthogonal to the euclidean n-flat, we emphasize that the $(n+1)$-dimensional vector space is itself not euclidean. This proof gave us all the linear reflections preserving the cone, there is one and only one in each flat of B^n. Therefore, *the linear reflections preserving the cone form a smooth manifold diffeomorphic to the canonical line bundle of* $\mathbb{R}P^{n-1}$: the flats through b constitute the $\mathbb{R}P^{n-1}$; and for the flats L constituting any small open subset U of this manifold we can choose a continuous normal direction; so, identifying the other flats of the ball parallel to these L's with their centres ℓ on these directed diameters, we obtain local trivializations $U \times (-c, +c)$. We note also that this diffeomorphism type stays put even for $c = \infty$, i.e., when we are talking of the space of all orthogonal reflections of the euclidean n-flat. The next result of this paragraph can also be sharpened: *there is a unique linear reflection of the cone which switches any given pair of distinct rays S' and S''.* For, if neither ray is the central ray S, conjugation with g, the reflection of the cone switching S' and S, gives us a bijective correspondence between reflections switching S' and S'' and those switching S and gS'', but the latter set is a singleton. And, regarding our definition of congruence for n-ball geometry which concluded this paragraph, we note that, *up to a homothety, any linear isomorphism of the cone is a composition of at most $n+1$ linear reflections of the cone.* For, if the isomorphism maps the central ray S to S', then by composing it, if need be, with the reflection switching $S \neq S'$ and a homothety we obtain a linear isomorphism of the cone which is the identity on S. It maps any diameter PQ of the n-ball to a line segment $P'Q'$ having b as its mid-point and with P' and Q' on the boundary of the cone, which is possible only if $P'Q'$ is also a diameter of the n-ball. So our map restricts to an isometry of the euclidean n-flat mapping b to itself, hence it is a composition of at most n orthogonal reflections. Also, *the linear reflections of the cone are conjugate to each other in the group $G(n)$ of all their compositions*: for, if $\ell \neq b$, then conjugation with the linear reflection of the cone which switches the rays through these two points gives us a reflection whose flat passes through b, etc.

11. Using notes 9 and 10, $G(n)$ *is isomorphic to the group of matrices A relating ordered pairs of framed observers.* The 'dictum' of the fourth paragraph says that if observers $S' \neq S$ are equipped with mirror image frames under the linear reflection of the cone switching them, then their measurements must be related by the corresponding matrix A. Since this is obviously true also if the same observer replaces the frame he is using by any orthogonal reflection, *the measurements made by any ordered pair of framed observers are related by the corresponding matrix A.* Usually the orientations of the framed observers are compatible with each other, so only those matrices A come into play whose determinants are positive, equivalently, only the subgroup $G_+(n)$ of all compositions of any *even number* of reflections is admitted. Therefore, as we mentioned before in note 5, mirror relativity is slightly stronger. Notably, *an invariant vector* $\mathbf{a} = \mathbf{0}$ *even for* $n = 1$, which is false if an orientation is preferred: the concluding step in the argument that we used to obtain the mass-energy formula is then valid only for $n \geq 2$. The subgroup $G_+(1)$ of translations is abelian, but the group $G(1)$ of all motions of a 1-ball (the real line has isomorphic groups) is not commutative. This homogenous geometry of a 1-ball, i.e., a bounded open interval, is however not non-euclidean per our usage of this adjective, because the parallel postulate is trivially true. Sometimes this adjective is used *only* for homogenous geometries, then of course it is not at all true that non-euclidean geometries are 'dime-a-dozen' for $n \geq 2$. As for the 'pragmatic considerations' of the fourth paragraph, *these objections were* (imho) *raised to his geometry by practical fellow Egyptians even in Euclid's lifetime!* A down-to-earth person is none too impressed by lines that don't end, or a parallelism of line segments that is not experimentally decidable: a bounded subset of Euclid's plane, above all a disk of a possibly large but finite radius around him, is eminently more reasonable to him. The quotation in this paragraph is from a paper by Arnol'd which is available on my website. It alerts us that, *it is the individual times* t' *and the euclidean spaces* $t' = 1$ *of our 'ball's worth' of observers* S' *that are basic,* what frame an experimenter uses to make his measurements, or a theoretician to do his calculations, is only of secondary importance.

12. A modicum of calculations, in the remainder of our 3-page essay, then gave us time dilation and length contraction by $\gamma(v) = (1-v^2/c^2)^{-1/2}$, the clock paradox, and the mass-energy formula $E = mc^2$. Also we saw that, *relativity is a hidden variables theory* : all the mirror images of any point of the cone form *an absolute but curved space,* and the homogenous function which is one on it *an absolute nonlinear time* τ; but to each observer S' this curved hidden space appears flat, a ball B' of radius c around him, on which his linear time $t' = 1$. The cone is the same for all the observers, but only its hidden foliation τ = constant is preserved by the full group $G(n)$ generated by all the linear reflections of the cone, its observed foliation into parallel balls t' = constant is preserved only by the subgroup $O(n)$ generated by the reflections preserving the observer S'. That a space consists of all the mirror images of any of its points is nice, but, *infinite divisibilty is not pragmatic* : one may object to Euclid's plane also on the grounds that it can be tiled by an *arbitrarily small* square! Magically, this new objection is also taken care of if we confine ourselves to an n-ball, $n \geq 2$, of radius $c < \infty$: *if a polytope of rays tiles absolute space, its riemannian volume is bigger than a positive constant depending inversely on* c. So the volumes of the closed manifolds B^n/Γ of note 3 are all more than this constant. An observer hears the hidden shapes of these 'particles' as proper values of Γ-periodic 'waves' on the covering space $\tau = 1$ whose differential equation can be written in his coordinates. Historically, *the theories of the subatomic world also arose from* $c < \infty$, but no one seems to understand this other side of the relativistic coin really well. Anyway, some of what little I myself have been able to understand about these quotients is in *Hyperbolic Manifolds* (2012), which will be available from my website as soon as I can write a prefatory note explaining what I was up to in this unfinished paper.

13. The cone is all the spacetime one really needs. Like positive numbers on the real line (cf. note 3) it is closed under $P+Q$, but all its differences $P-Q$ form the full $(n+1)$-dimensional vector space. The *partial order* defined by, $P > Q$ iff $P-Q$ is in cone, is quite basic : *cartesian absolute motions are precisely all the directed and strictly increasing smooth arcs in the cone.* For $P > Q$ implies $\tau(P) > \tau(Q)$ – the converse is not true for $c < \infty$ – and that extra condition '$d\mathbf{r}/d\tau$ nonzero along a ray $S'(\tau)$' on the smooth arc is equivalent to saying that if P comes after Q, then QP is parallel to a ray of the cone, i.e., $P > Q$. Also, *it is true that* $P > Q \iff \tau(P+R) > \tau(Q+R)\ \forall\ R$, but in our set-up parallel motions are deemed to be the same, therefore, *if we admit only the irreversibility of time, then these are all the possible smooth motions.*

14. Considering what all had gone into that definition – see note 6 – of the riemannian metric on $\tau = 1$, it is a miracle that *the associated pseudometric on the cone makes sense for any bounded open convex subset* Ω^n *of affine n-space!* The distance $\widehat{AB}$ between the rays through A and B is equal to $\frac{c}{2}\log(\frac{XB}{XA}\frac{YA}{YB})$ if AB extended meets the boundary in X and Y. In this *two-line Ph. D. thesis*—as Littlewood dubbed this discovery—*of Cayley's,* $c > 0$ is arbitrary, but if Ω^n is an open ball of a norm $\|\cdot\|$ on n-space, e.g. $\Omega^n = B^n$, the best choice is its radius. For then this definition also gives us a distance between rays through B^n which is preserved by all the reflections of the cone, *and* which approaches the euclidean distance for $c \to \infty$, so it coincides with the riemannian metric. To check this we'll again, as in Figure 2 and the subsequent paragraph, temporarily think of the $(n+1)$-dimensional vector space as euclidean.

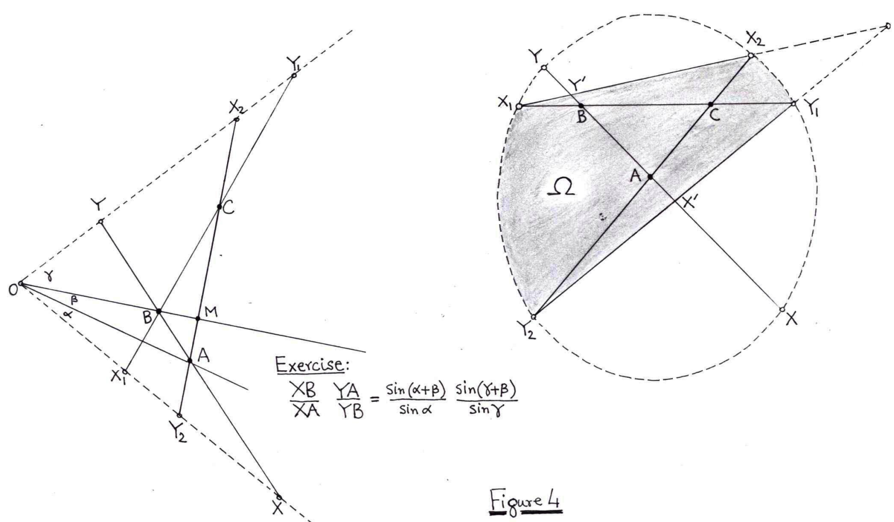

Figure 4

Though the ratios $\frac{XB}{XA}$ and $\frac{YA}{YB}$ depend on the line cutting four given coplanar and coincident lines in X, A, B and Y (unless they are parallel, i.e., coincide at infinity) their product $\frac{XB}{XA}\frac{YA}{YB}$ *is an invariant.* Indeed, using the sine law for triangles one can check – Figure 4, Exercise – that in this *biratio* one can replace each length by the sine of the subtended angle. Using this invariance we'll now prove the *triangle inequality* $\widehat{AC} + \widehat{CB} \geq \widehat{AB}$ whenever extending each side gives us two points on the boundary. When ABC is in a plane through the origin 0, one side is in fact *equal* to the sum of the other two, for example, for the triangle drawn in Figure 4, $\widehat{AB} + \widehat{BC} = \widehat{AM} + \widehat{MC} = \widehat{AC}$. If 0 is not in the plane of ABC, a similar argument gives us $\widetilde{AC} + \widetilde{CB} = \widetilde{AB}$ *for the convex open planar subset* Ω, shown shaded in Figure 4, of the cone between the (possibly parallel) lines X_1X_2 and Y_2Y_1. Which implies the desired inequality, for the left side is the same as $\widehat{AC} + \widehat{CB}$, and we have $\widetilde{AB} \geq \widehat{AB}$ because $\frac{X'A}{X'B} \geq \frac{XA}{XB}$ and $\frac{Y'A}{Y'B} \geq \frac{YA}{YB}$. Any linear reflection preserving the cone preserves its Cayley distance because it does so on the ellipsoidal section – see Figure 2 – on which it coincides with an orthogonal reflection. Finally, we note that a point of the n-ball at euclidean distance r from its centre is at Cayley distance $\frac{c}{2}\log(\frac{c+r}{c-r})$, and this quantity approaches r as $c \to \infty$.

15. By a *piecewise linear absolute motion* $P_0P_1 \ldots P_k$ we mean a directed and strictly increasing—in the sense of note 13—broken line in the cone. *The elapsed time for this motion is* $\tau(\overrightarrow{P_0P_1}) + \tau(\overrightarrow{P_1P_2}) + \cdots + \tau(\overrightarrow{P_{k-1}P_k})$. This because, each $\overrightarrow{P_iP_{i+1}}$ is parallel to some ray S', on which ray τ coincides with the linear time t' of this galilean observer, so $\tau(\overrightarrow{P_iP_{i+1}}) := \tau(P_{i+1} - P_i) = t'(P_{i+1} - P_i) = t'(P_{i+1}) - t'(P_i)$ is the time recorded by the moving clock over this segment. However, *since the absolute time τ is non-linear for $c < \infty$,* we can't write $\tau(P_{i+1} - P_i) = \tau(P_{i+1}) - \tau(P_i)$, and then cancel etc., to get $\tau(P_k) - \tau(P_0)$. Instead, *we have the startling* clock paradox *:* $\tau(\overrightarrow{P_0P_1}) + \tau(\overrightarrow{P_1P_2}) + \cdots + \tau(\overrightarrow{P_{k-1}P_k}) \leq \tau(P_k) - \tau(P_0)$, with equality iff the P_i's are all on the same ray. Equivalently, *the reversed triangle inequality* $\tau(\overrightarrow{AC}) + \tau(\overrightarrow{CB}) \leq \tau(\overrightarrow{AB})$ *holds,* for any three points $0 \leq A < C < B$, with equality iff they are collinear. To see this recall that in the sixth paragraph of the text we showed that $\tau^2(t'; \mathbf{x}') = t'.t' - \frac{1}{c^2}\mathbf{x}'.\mathbf{x}'$ if one uses components parallel to the time and the euclidean space of any galilean observer S'. If we take S' parallel to $\overrightarrow{AB}$, then $\mathbf{x}'(A) = \mathbf{x}'(B)$, so the right side is $t'(B) - t'(A)$. Let M be the point on $\overrightarrow{AB}$ such that $t'(M) = t'(C)$. Then $\tau^2(\overrightarrow{AC}) = (t'(M) - t'(A))^2 - \frac{1}{c^2}\overrightarrow{MC}.\overrightarrow{MC} \leq (t'(M) - t'(A))^2$, with equality iff $M = C$. Likewise $\tau(\overrightarrow{CB}) \leq t'(B) - t'(M)$, which completes the proof. We note that, if the above absolute motion is given by the vector function $\mathbf{r}(s)$ of elapsed time, then on the interior of each $\overrightarrow{P_iP_{i+1}}$ we have $ds = dt'$ and $\frac{d\mathbf{r}}{ds} = \overrightarrow{0b'}$, the 'absolute velocity' of the parallel S'. So the above definition of elapsed time is the same as in the seventh paragraph of the text, only then we looked at directed and strictly increasing arcs that are *smooth,* i.e., which are, so to speak, broken lines with infinitely many infinitesimally small links $d\mathbf{r}$. Instead of the above finite sum, it is the analogous riemann integral $\int_{P_0}^{P} \tau(d\mathbf{r})$ taken along the motion that gives us the elapsed time $s(P)$ till any point, so $\tau(d\mathbf{r}) = ds$, i.e., $\tau(\frac{d\mathbf{r}}{ds}) = 1$. That is, the length of $\frac{d\mathbf{r}}{ds}$ is identically 1 with respect to the quadratic form τ^2; so, as in euclidean differential geometry, we'll also call this derivative the *unit tangent vector* $\mathbf{u}(s)$ at any point of the smooth motion.

16. *For $c < \infty$ extra hypotheses like smooth or p.l. are not needed!* A strictly increasing function from an interval into the cone is trapped near each point in the parallel cone, so it is continuous and, *as seen by any observer S in his euclidean space this à priori motion is lipschitz in his time t with constant c,* i.e. $\|\mathbf{x}(t_1) - \mathbf{x}(t_0)\| < c|t_1 - t_0|$, so it is differentiable almost everywhere. The same integral gives the elapsed time $s(P)$, and this motion has a unit tangent vector $\frac{d\mathbf{r}}{ds}$ a.e., but $\frac{d^2\mathbf{r}}{ds^2}$ is only a generalized function or distribution. For example, for a p.l. motion it is supported on the finitely many bends. Nevertheless, *the equations around the mass-energy formula in the eighth paragraph of the text are still valid weakly.* Likewise, the yang-mills formulary, which depends on the special feature of 4-dimensional space that $SO(4)$ is not simple, is valid weakly if we allow all these motions, which suffices to deduce that, *there exist topological 4-manifolds which do not admit any lipschitz structure!* On the other hand in Sullivan, *Hyperbolic geometry and homeomorphisms* (1979), it was shown that, *any topological n-manifold, $n \neq 4$, has a unique lipschitz structure!* At one point in this almost surreal paper – it is available on my website – Sullivan invokes his paper with Deligne that was cited in note 3.

17. *Arbitrarily close to any à priori motion is a piecewise linear absolute motion with the same end points and with elapsed time arbitrarily small!* A riemann sum involves an approximating broken line with almost the same elapsed time; to make this time arbitrarily small use the fact that, *any two points on a ray can be joined by a planar zig-zag of a small amplitude whose links are alternately almost parallel to the two boundary rays.* Smooth absolute motions, even those with a small elapsed time, are likewise dense in à priori motions. The clock paradox is less startling when stated thus: *a journey takes the maximum elapsed time if no force is expended.* The time-stopping oscillations above have impractically big accelerations, perhaps these should be banned too by *a new decree?* We showed in the eighth paragraph of the text $\frac{d^2\mathbf{r}}{ds^2} \times \frac{d\mathbf{r}}{ds} = 0$, i.e., the rate of change of the absolute velocity $\frac{d\mathbf{r}}{ds} = \overrightarrow{0b'}$ is constantly orthogonal to it with respect to the quadratic form τ^2. That is, if we draw an arrow parallel and equal in length to $\frac{d^2\mathbf{r}}{ds^2}$ from b', then it is contained in the euclidean space $t' = 1$. It seems reasonable to us that this arrow should be confined to a ball of a prescribed radius around b'. Which radius, by changing units, we can take once again to be c itself. So, *we can decree that the absolute acceleration $\frac{d^2\mathbf{r}}{ds^2}$ should always remain in the balls B'.* Under this decree, there is a positive lower limit on the elapsed times of journeys between two events.

18. In the sixth paragraph of the text we measured vectors parallel to lines cutting the boundary twice by applying $\tau^* := \sqrt{-c^2\tau^2}$: *it too does not obey the triangle inequality.* If P and Q are two points on any such line, and we draw through them, in the plane containing 0, lines parallel to the two boundary rays, then a path PRQ in this parallelogram and close to its boundary has in fact an arbitrarily small $\tau^*(PR) + \tau^*(RQ) < \tau^*(PQ)$. However, τ^* *on vectors lying in any ball B' gives their euclidean length,* for $-c^2\tau^2(t';\mathbf{x}') = -c^2t'.t' + \mathbf{x}'.\mathbf{x}'$ if one uses components parallel to the time and the euclidean space of S'. This is the relative speed—always less than $2c$—between observers as observed by S', and this distance between rays is invariant under reflections of the cone preserving

S'. It is τ^* on point-pairs of *that hidden space* $\tau = 1$ *enveloped by all these balls,* that is $\tau^*(b'b'')$, that gave us an observer-independent and fully invariant proper speed between observers; and $\inf \int_{b'}^{b''} \tau^*(d\mathbf{r})$ over all curves on $\tau = 1$ from b' to b'' gives – note 6 – an invariant distance between rays obeying the triangle inequality. In note 14 we showed that this must be Cayley's distance: *if* S'' *has speed* v *as observed by* S', *then* $\inf \int_{b'}^{b''} \tau^*(d\mathbf{r}) = \frac{c}{2}\log(\frac{c+v}{c-v})$, *with* inf *attained on and only on the curve from* b' *to* b'' *on* $\tau = 1$ *and* $\mathrm{span}\{S', S''\}$. To double-check this let $\frac{c}{2}\log(\frac{c+v}{c-v}) = c\theta$, then $v = c\tanh\theta$, but $\mathrm{span}\{S', S''\} \cap \{\tau = 1\}$ has in the coordinates (t', x') of S' the cartesian equation $-c^2t'^2 + x'^2 = -c^2$ or parametric equations $t' = \cosh\theta, x' = c\sinh\theta$, so over this curve $\int_{b'}^{b''} \tau^*(d\mathbf{r}) = \int_0^\theta \sqrt{-c^2(dt')^2 + (dx')^2} = c\theta$. Also, *the retraction* $P = (t'; \mathbf{x}') \mapsto (t', \|\mathbf{x}'\|) = \overline{P}$ *of the vector space on the half-plane* $x' \geq 0$ *of* $\mathrm{span}\{S', S''\}$ preserves $\tau = 1$ and, for any point-pair on it, since $\|\mathbf{x}'(PQ)\| \geq |\|\mathbf{x}'(P) - \|\mathbf{x}'(Q)\||$, we have $\tau^*(PQ) \geq \tau^*(\overline{PQ})$, with equality iff $\mathbf{x}'(P)$ or $\mathbf{x}'(Q)$ is a non-negative multiple of the other. Hence $\int_{b'}^{b''} \tau^*(d\mathbf{r}) \geq \int_{b'}^{b''} \tau^*(d\overline{\mathbf{r}})$ for any $\mathbf{r}(u)$ on $\tau = 1$ from b' to b'' and its retraction $\overline{\mathbf{r}}(u)$, with equality iff $\mathbf{r}(u) = \overline{\mathbf{r}}(u)\forall u$. *q.e.d.* Here of course, following Riemann, $\int_{b'}^{b''} \tau^*(d\mathbf{r}) := \lim[\tau^*(P_0P_1) + \tau^*(P_1P_2) + \cdots + \tau^*(P_{k-1}P_k)]$, as one takes more and more closely spaced points $b' = P_0, P_1, \ldots, P_k = b''$ in order on $\mathbf{r}(u)$. For the minimizing curve $\mathbf{r}(\theta)$, which is on the plane through 0, $\tau^*(P_iP_{i+1}) = c\sqrt{2\cosh(\theta_{i+1} - \theta_i) - 2} > c(\theta_{i+1} - \theta_i)$, so now its riemann sums are bigger than the integral, but steadily decrease to it under refinement.

19. To elaborate on note 7 we'll switch to $(t_1; \mathbf{x_1})\star(t_1; \mathbf{x_2}) = -c^2t_1t_2 + \mathbf{x_1}.\mathbf{x_2}$, the bilinear form associated to $-c^2\tau^2$. So, if $\mathbf{r}(s)$ is any smooth à priori motion parametrized by elapsed time and $\mathbf{u}(s) = \frac{d\mathbf{r}}{ds}$ is its unit tangent vector field – note 15 – then $\mathbf{u}(s) \star \mathbf{u}(s) = -c^2$. The $\star$-orthogonal complement of $\mathbf{u}(s)$ is the euclidean space of the parallel galilean observer and on it our bilinear form coincides with its dot product. So one has moving *frames* $\{\mathbf{u}(s); \mathbf{e_1}(s), \ldots, \mathbf{e_n}(s)\}$ of smooth vector fields along the motion such that $\mathbf{e_i}(s) \star \mathbf{e_j}(s) = \delta_{ij}$ and $\mathbf{u}(s) \star \mathbf{e_i}(s) = 0$. An à priori motion is a parallel pencil of directed and strictly increasing arcs – note 13 – in the cone. By perturbing such an arc, in the interior of the smooth manifold $a \leq \tau \leq b$ on whose boundary its end points lie, we can replace it by another which is lipschitz close to it, and which is not only smooth but also *generic,* i.e., its first $n + 1$ derivatives are always linearly independent. A smooth generic motions has a *frenet frame* : each unit vector $\mathbf{e_i}(s), 1 \leq i \leq n$, is obtained by multiplying the component of $\frac{d^{i+1}\mathbf{r}}{ds^{i+1}}$ orthogonal to $\mathrm{span}\{\mathbf{u}(s), \mathbf{e_1}(s), \ldots, \mathbf{e_{i-1}}(s)\}$ with the reciprocal of its nonzero length $\kappa_i(s)$. Since adding a vector does not change derivatives these *curvatures* $\kappa_i(s)$ are well-defined, a parallel arc is also generic with the same elapsed times and curvatures at its corresponding points. Moreover, *two smooth generic motions are related by a finite sequence of linear reflections of the cone if and only if they have the same curvature functions* $\kappa_i(s), 1 \leq i \leq n$. Any linear transformation L of the cone preserves τ and $\star$ and maps a smooth generic motion onto another whose derivatives are the images under L of its derivatives; this shows 'only if'. For 'if' represent the two motions by arcs whose initial points are on rays parallel to their initial unit tangent vectors, and then slide one of these arcs along this ray so that τ has the same value on both initial points. Now apply the linear reflection of the cone interchanging these two rays to one of the arcs to get two arcs with the same initial point and the same $\mathbf{u}(0)$ along the ray through this point. Finally, by applying to one of the arcs the orthogonal transformation of this observer which throws the initial unit vectors $\mathbf{e_i}(0)$ of this arc onto those of the other we are reduced to showing that, *there exists a unique smooth generic arc parametrized by elapsed time with a given initial frenet frame* $\{\mathbf{u}(0); \mathbf{e_1}(0), \ldots, \mathbf{e_n}(0)\}$ *and given curvature functions* $\kappa_i(s)$. For this we'll need the *frenet equations* : $\frac{d\mathbf{u}}{ds} = k_1(s)\mathbf{e_1}(s)$, $\frac{d\mathbf{e_i}}{ds} = k_{i+1}(s)\mathbf{e_{i+1}}(s) - k_i(s)\mathbf{e_{i-1}}(s)$ for $1 \leq i < n$ and $\frac{d\mathbf{e_n}}{ds} = -k_n(s)\mathbf{e_{n-1}}(s)$, where $k_1(s) = \kappa_1(s)$, $\mathbf{e_0}(s) = \mathbf{u}(s)$ and $k_{i+1}(s) = \frac{\kappa_{i+1}(s)}{\kappa_i(s)}$. The first equation is 'what gave $E = mc^2$', viz., $\frac{d^2\mathbf{r}}{ds^2} \star \frac{d\mathbf{r}}{ds} = 0$, i.e., derivative of $\mathbf{u}(s) \star \mathbf{u}(s)$ is zero, i.e., $\mathbf{u}(s) \star \mathbf{u}(s)$ is constant. Likewise $\mathbf{e_i}(s) = \frac{1}{\kappa_i(s)} \frac{d^{i+1}\mathbf{r}}{ds^{i+1}}$ + lower order terms, has derivative $\frac{1}{\kappa_i(s)} \frac{d^{i+2}\mathbf{r}}{ds^{i+2}}$ + lower order terms, which for $1 \leq i < n$ is equal to $\frac{\kappa_{i+1}(s)}{\kappa_i(s)}\mathbf{e_{i+1}}(s)$ + a linear combination of $\mathbf{e_j}(s)$ *with* $j \leq i$ *only,* while for $i = n$ even the leading term is missing. Now use $\frac{d\mathbf{e_i}}{ds} \star \mathbf{u}(s) + \frac{d\mathbf{u}}{ds} \star \mathbf{e_i}(s) = 0$ and $\frac{d\mathbf{e_i}}{ds} \star \mathbf{e_j}(s) + \frac{d\mathbf{e_j}}{ds} \star \mathbf{e_i}(s) = 0$ – these express the constancy of $\mathbf{e_i}(s) \star \mathbf{u}(s)$ and $\mathbf{e_i}(s) \star \mathbf{e_j}(s)$ – to obtain the other n equations. These $n+1$ equations can be written more compactly as the *matrix equation,* $\frac{dU}{ds} = K(s)U(s),$ where $U(s)$ is the square matrix of size $n+1$ with row vectors $\mathbf{u}(s), \mathbf{e_1}(s), \ldots, \mathbf{e_n}(s)$, and $K(s)$ is the skewsymmetric matrix of this size whose only nonzero entries are $k_1(s), \ldots, k_n(s)$ immediately above the main diagonal, and their negatives below it. Using the existence and uniqueness theorem for linear ODE's, this equation has one and only one solution $U(s)$ with initial value $U(0)$, and its first row $\mathbf{u}(s)$ determines the motion $\mathbf{r}(s) = \mathbf{r}(0) + \int_0^s \mathbf{u}(s)ds$. This solution $U(s)$ is *not* in general given by $U(s)U(0)^{-1} = \exp \int_0^s K(s)ds$ but this formula is true if the curvatures are constant. However, *any n smooth positive functions can be realized as the curvatures of a smooth generic motion,* unless some new decree – note 17 – on acceleration and higher derivatives is in force. À priori motions also contain another open dense set, of piecewise linear motions with vertices in general position, and, *there is a similar classification of generic p.l. motions.* This frenet theory is akin to Kalai's *algebraic shifting,* a simple but surprisingly useful idea, over which I had mulled for long in the 1990's, but never quite managed to grasp it to my satisfaction ...

20. From the geographical distribution of some traits in the mitochondrial DNA sequence it has been deduced that, *all women are the iterated daughters of just one, who lived in Africa 150,000 years ago!* So once upon a time, whatever mathematics there was, was in Africa only. The recorded history of our subject is shorter, but Africa looms large in it too, in particular, the school of mathematics that flourished in Alexandria—for more than six hundred years!—from Euclid to Pappus. Practically everything above is rooted in those books of the former, and the invariance of the biratio – see note 14 – is only one of the many things about projective geometry that can be found in the prolific writings of the latter. The log put by Cayley before it merely converted this multiplicative distance into an additive one, and may even turn out to be a retrograde step, when we switch to other fields to understand the galois symmetries alluded to in notes 1 and 3. The sine law for triangles, and much else from plane and spherical trigonometry, was known to Ptolemy if not Heron, both of Alexandria. Also, the latter knew that light travels so that minimum time is taken, and had used this to prove the isoperimetric inequality : in fact you'll recall that he once gave a colloquium talk (!) on this very topic even in imperial Rome, see *Extracts* from my Notebooks (2008). Mathematics teaches us humility : much repetition and reworking, sometimes stretching over centuries, is often needed before a key, but in hindsight obvious, idea sinks in. It is then mostly a matter of personal choice as to which mathematician, or a set of mathematicians, one wants to credit with this idea. Though the definition of 'his' distance is all there in a rambling 1859 paper of Cayley's, the crisp 2-line format owes much to Beltrami and Klein, and one presumes that during these years it became 'folklore' that it assigns a length to the segments of any bounded open Ω^n, but this came out only when a letter from Hilbert to Klein was published in 1895. Hilbert was trying to make Euclid's formal presentation of geometry more rigorous, and he tells Klein excitedly that there is a bounded open convex set—to wit the shaded region in Figure 4—with more than one geodesic between two points! *Cayley's*—or if you prefer Pappus's or Lobatchevsky's or Riemann's or Beltrami's or Klein's or Hilbert's ...—*distance breathes life into all the 'dime-a-dozen' geometries that we mentioned at the very outset,* and then again in note 1. Some more water has flown north out of Africa past Alexandria since Hilbert included in his famous list of problems *two* that were closely related to this letter. So much more is known now, for example, Benoist has characterized those convex Ω^n whose dime-a-dozen geometry is hyperbolic in the sense of Gromov. But much still remains to be done, even for convex polytopes ... Moreover, *each of these geometries comes with a concomitant linearization or relativity theory.* Indeed, the definition of an unparametrized cartesian absolute motion is already clear from note 13, and absolute space can be defined once again to be the envelope of all the images of Ω^n under the—now possibly very few—linear symmetries of its cone, so absolute time, et cetera. The paucity of symmetries can be converted from a handicap to a boon, for example, one can focus better on some subgroups of G by replacing the ball itself by a symmetric polytope, and it seems galois symmetries will make up for some loss of symmetry too ...

K S Sarkaria

(*contd.*)