

KURATOWSKI COMPLEXES

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§1. INTRODUCTION

An n -dimensional simplicial complex K , $n \geq 1$, which does not embed in \mathbb{R}^{2n} , is said to be *minimally non-embeddable* in \mathbb{R}^{2n} if all proper subspaces of the same are embeddable in \mathbb{R}^{2n} . The well known planarity criterion of Kuratowski [6], 1930, tells us that, for $n = 1$, there are, upto homeomorphism, only two such simplicial complexes, σ_1^1 and σ_0^2, σ_0^2 . (Here, and below, σ_j^i denotes the j -skeleton of an i -simplex, and, while taking joins, it is understood that disjoint copies of the simplicial complexes are being used.) However, for $n \geq 2$, it is easy to see that there exist infinitely many non-homeomorphic simplicial complexes, all minimally non-embeddable in \mathbb{R}^{2n} . The object of this note is to show that a higher-dimensional analogue of Kuratowski's theorem is valid, provided minimality is replaced by an (at least for $n \neq 2$) stronger notion.

An n -dimensional simplicial complex K , $n \geq 1$, which does not embed in \mathbb{R}^{2n} , will be called *critically non-embeddable* in \mathbb{R}^{2n} if, for each pair (x, y) of points lying in some pair of disjoint simplices of K , one can find a continuous map $f_{x,y}: K \rightarrow \mathbb{R}^{2n}$ which has no other such pair of points as a double point; and moreover, locally, these maps $f_{x,y}$ can be chosen to depend continuously on the parameters (x, y) .

We recall that the *deleted join* K_* of a simplicial complex K is the subcomplex of $K.K$ (i.e. of ${}^1K.{}^2K$, the join of two disjoint copies 1K and 2K of K ; the simplices of $K.K$ are usually denoted (σ, θ) rather than ${}^1\sigma \cup {}^2\theta$) consisting of all simplices (σ, θ) such that $\sigma \cap \theta = \phi$.

THEOREM (1.1) *An n -dimensional simplicial complex K , $n \geq 1$, $n \neq 2$, is (a) critically non-embeddable in \mathbb{R}^{2n} iff (b) its deleted join K_* is Z_2 -homeomorphic to the antipodal $(2n + 1)$ -sphere iff (c) K_* is a homogenously $(2n + 1)$ -dimensional pseudomanifold iff (d) K is a join of some simplicial complexes of the type $\sigma_{s_i-1}^{2s_i}$, $s_i \geq 1$.*

Thus there are, upto simplicial isomorphism, $\pi(n + 1)$ such *Kuratowski n -complexes*, $\sigma_{s_1-1}^{2s_1} \cdot \sigma_{s_2-1}^{2s_2} \cdot \dots \cdot \sigma_{s_k-1}^{2s_k}$, one for each partition of $n + 1$ as a sum $s_1 + s_2 + \dots + s_k$ of positive integers, $1 \leq s_1 \leq s_2 \leq \dots \leq s_k \leq n + 1$.

The purely combinatorial implication (c) \Rightarrow (d) will be established in §2. The rest of the proof, which essentially depends on the ideas contained in van Kampen [15], 1932, and Flores [2], 1933, will be given in §3.

Though our argument yields (a) \Rightarrow (c) only under the condition $n \neq 2$, the above result is probably true for all $n \geq 1$.

(1.2) *Notation.* Since the context (e.g. the mention of a simplicial, continuous, or linear map) leaves little room for confusion, we will usually employ the same letter K when

referring to a combinatorial *simplicial complex*, i.e. a finite set K of finite sets obeying $\sigma \in K, \theta \subseteq \sigma \Rightarrow \theta \in K$, or to a topological or geometrical realization thereof. Likewise, depending on the context, the same letter σ will stand for an abstract *simplex*, i.e. a finite set of *vertices*, or for a *closed simplex* occurring as a topological or geometrical realization thereof. (In addition, in the context of integral (co) chains, it is understood that σ carries an *orientation*.) The elements of the closed simplex are called the *points* of σ . If one excludes from σ those points which belong to the proper faces of σ , then one gets the *open simplex* associated to σ ; this will be denoted by $\text{int } \sigma$. The *star*, *closed star*, and *link* of σ in K are defined in the usual way, and denoted by $St_K \sigma, \overline{St}_K \sigma$, and $Lk_K \sigma$ (notation to be interpreted combinatorially, topologically, etc., depending on the context). And, as usual, superscripts will often be used to indicate dimensions.

Two simplices will be called *disjoint* iff their sets of vertices are disjoint, otherwise they are called *adjacent*. The set of vertices of K will be denoted by V_K (or just V) while the subset of vertices occurring in the link (resp. the closed star) of σ will be denoted $L_K \sigma$ (resp. $S_K \sigma$). The cardinality of $L_K \sigma$ is also called the *valence* of σ . A subset $c \subseteq V_K$ is called a *circuit* of K if $c \notin K$ but all proper faces of c belong to K . A *full subcomplex* of K is one which contains all simplices of K whose vertices are in the subcomplex. A simplicial complex is determined by its maximal or *basic* simplices. If all of these are of the same dimension, then K is called dimensionally *homogeneous*. If further any codimension one simplex is incident to precisely 2 basic simplices, then K is called a *pseudomanifold*.

As usual the *join* $K \cdot L$ will be defined only for disjoint simplicial complexes, and consists of all simplices $\sigma \cup \theta, \sigma \in K, \theta \in L$. Quite often disjoint unions $\sigma \cup \theta$ will also be written (σ, θ) or $\sigma \cdot \theta$ or $v_1 \dots v_k \cdot \theta$ (where v_i are the vertices of σ) etc. Recall that as a space $K \cdot L$ is made up of the points of the line segments having one end in K and the other in L . Joins of oriented simplices or spaces will be equipped with the orientation of the first factor "followed" by that of the second. On joins $K \cdot K$ of disjoint copies of the same simplicial complex one has the simplicial Z_2 -action $(\sigma, \theta) \mapsto (\theta, \sigma)$. A Z_2 -(co)chain of a Z_2 -simplicial complex will be one which is preserved upto sign by its Z_2 -action. An *elementary* Z_2 -cochain, i.e. one which can be nonzero only on one pair of oriented simplices, will be specified by the value it takes on one member of this pair.

§2. CLASSIFICATION

Proof of (c) \Rightarrow (d). Let K denote an n -dimensional simplicial complex for which K_* is a $(2n + 1)$ -pseudomanifold.

(2.1) K must be homogeneously n -dimensional, with each $(n - 1)$ -simplex incident to at least 3, and at most $n + 3$, n -simplices. Further, K can have an $(n - 1)$ -simplex incident to $n + 3$ n -simplices only if K is the n -skeleton of a $(2n + 2)$ -simplex.

Given any $\alpha \in K$, the fact that K_* is homogeneously $(2n + 1)$ -dimensional gives us a $(\omega^n, \varphi^n) \in K_*$ containing (α, ϕ) . Thus $\alpha \in K$ is contained in some $\omega^n \in K$, i.e. K too is dimensionally homogeneous. Next, given any $\sigma^{n-1} \in K, v \in L_K \sigma^{n-1}$, choose a $(\sigma^{n-1}, \omega^n) \in K_*$ containing (σ^{n-1}, v) . In order that this $2n$ -simplex of K_* be incident to two $(2n + 1)$ -simplices of K_* it is necessary that there be at least two other vertices in $L_K \sigma^{n-1}$. On the other hand if $L_K \sigma^{n-1}$ had more than $n + 3$ elements, then 3 or more of these would be outside ω^n , and so (σ^{n-1}, ω^n) would have valence bigger than 2. Finally, let $L_K \sigma^{n-1}$ have exactly $n + 3$ elements, and let w be any vertex of K . If w is not in σ^{n-1} , then it must be in $L_K \sigma^{n-1}$; otherwise a $(\sigma^{n-1}, \varphi^n) \in K_*$ containing (σ^{n-1}, w) would have valence > 2 . Thus K is a subcomplex of τ_n^{2n+2} , the

n -skeleton of the $(2n+2)$ -simplex τ^{2n+2} formed by the $2n+3$ vertices of $S_K \sigma^{n-1}$. Any two $(2n+1)$ -simplices of $(\tau_n^{2n+2})_*$ can be "joined" to each other via a sequence of $(2n+1)$ -simplices, each sharing a $2n$ -face with the preceding. Hence $K_* \subseteq (\tau_n^{2n+2})_*$ can be a pseudomanifold only if $K_* = (\tau_n^{2n+2})_*$ and so $K = \tau_n^{2n+2}$.

(2.2) For all $\sigma^{n-1} \in K$, $\theta^n \in K$ and $\lambda \in L_K \sigma^{n-1} \setminus \theta^n$, the simplex $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus \lambda)$ lies in K .

We will prove this by induction on the cardinality of $\sigma^{n-1} \cap \theta^n$.

If σ^{n-1} is disjoint from θ^n , then θ^n contains all but 2 of the vertices of $L_K \sigma^{n-1}$; otherwise $(\theta^n, \sigma^{n-1}) \in K_*$ has valence $\neq 2$. Thus $L_K \sigma^{n-1} \setminus \theta^n \setminus \lambda$ consists of a single vertex $\mu \in L_K \sigma^{n-1}$, and we are merely saying that $\sigma^{n-1} \cdot \mu \in K$.

To prove the inductive step choose any $v \in \sigma^{n-1} \cap \theta^n$, and then a $w \in V$, if possible from outside $L_K \sigma^{n-1}$, in such a way that $\xi^n = w \cup (\theta^n \setminus v)$ is an n -simplex of K disjoint from $v \cup (\sigma^{n-1} \setminus \theta^n)$; this can be done because K_* is dimensionally homogenous (we'll use this fact repeatedly below also). Note that $\sigma^{n-1} \setminus \xi^n \setminus v = \sigma^{n-1} \setminus \theta^n$. If $w \notin L_K \sigma^{n-1}$ then $(L_K \sigma^{n-1} \setminus \xi^n) = (L_K \sigma^{n-1} \setminus \theta^n)$; so the result follows immediately from the inductive hypothesis $(\sigma^{n-1} \setminus \xi^n) \cdot (L_K \sigma^{n-1} \setminus \xi^n \setminus \lambda) \in K$. If $w \in L_K \sigma^{n-1}$ then for any $\lambda \in L_K \sigma^{n-1} \setminus \theta^n$ one has $\lambda \cup (\theta^n \setminus v) \in K$: if not, $\lambda \neq w$ belongs to $L_K \sigma^{n-1} \setminus \xi^n = (L_K \sigma^{n-1} \setminus \theta^n) \setminus w$, and any $2n$ -simplex $(\theta^n \setminus v, \varphi^n)$ of K_* containing $(\theta^n \setminus v, (\sigma^{n-1} \setminus \xi^n) \cdot (L_K \sigma^{n-1} \setminus \xi^n \setminus \lambda))$ would have valence ≤ 1 . Next, let ω_μ^n be an n -simplex of K which is disjoint from θ^n and which contains the simplex $(\sigma^{n-1} \setminus \xi^n \setminus v) \cdot (L_K \sigma^{n-1} \setminus \xi^n \setminus \mu)$, $\mu \in L_K \sigma^{n-1} \setminus \xi^n$. Since K_* is a pseudomanifold we can further arrange that $\mu \notin \omega_\mu^n$ (or else that $w \notin \omega_\mu^n$). We assert that $w \in \omega_\mu^n$ (or else that $\mu \in \omega_\mu^n$): otherwise since $\theta^n \setminus v$ has all the vertices of $L_K \sigma^{n-1} \setminus \theta^n$, and also v , in its link, $(\theta^n \setminus v, \omega_\mu^n)$ would have valence ≥ 3 . Within all such simplices $\omega_\mu^n \in K$ we have thus all simplices of the type $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus \lambda)$, $\lambda \in L_K \sigma^{n-1} \setminus \theta^n = (L_K \sigma^{n-1} \setminus \xi^n) \cup w$.

As a corollary of (2.2) we see that

(2.3) K consists of all the independent sets of a matroid M_K on its set of vertices V_K .

By definition this means that the following is true.

(1) *Base Exchange Axiom.* $\forall \theta^n \in K, \omega^n \in K, u \in \omega^n \setminus \theta^n \exists v \in \theta^n \setminus \omega^n$ s.t. $(\omega^n \setminus u) \cup v \in K$.

If no such v were to exist, then, by applying (2.2) to θ^n and $\sigma^{n-1} = \omega^n \setminus u$, we see that $L_K \sigma^{n-1} \setminus \theta^n \setminus \lambda = L_K \sigma^{n-1} \setminus \lambda$ belongs to K . But this is not possible because then any $2n$ -simplex (σ^{n-1}, ω^n) of K_* containing $(\sigma^{n-1}, L_K \sigma^{n-1} \setminus \lambda)$ would have valence ≤ 1 .

We recall (see e.g. Welsh [17], pp. 13-15) that (1) is equivalent to saying that

(2) all full subcomplexes of K are dimensionally homogenous, i.e., that K obeys the following

(3) *Augmentation Property.* If $\alpha \in K, \beta \in K$ and $\dim \alpha < \dim \beta$, then α can be augmented to a $(\dim \beta)$ -dimensional simplex, $\alpha \cdot \gamma \in K$, by means of some $\gamma \subseteq \beta$.

Also (1) implies ([17], p. 68) that the circuits of K obey the following

(4) *Transitivity Property.* If the vertex u belongs to a circuit containing the vertex v , and v belongs to a circuit containing the vertex w , then u belongs to a circuit containing w .

As another application of (2.2) we now determine the exact nature of the circuits of K :

(2.4) A subset of V_K is a circuit of K iff it is of the type $L_K \sigma^{n-1} \setminus \lambda$ for some $\sigma^{n-1} \in K, \lambda \in L_K \sigma^{n-1}$.

While proving the base exchange axiom (2.3.1) we checked that $L_K \sigma^{n-1} \setminus \lambda \notin K$. On the other hand, by applying (2.2) to $\sigma^{n-1} \in K$ and $\theta^n = \lambda \cdot \sigma^{n-1} \in K$, we see that all proper subsets of $L_K \sigma^{n-1} \setminus \lambda$ belong to K . Conversely, given any circuit c of K , choose an n -simplex

$v \cdot \sigma^{n-1} \in K$ containing a vertex v of c , and disjoint from the simplex $c \setminus v \in K$ formed by the remaining vertices of c . Note that c is disjoint from σ^{n-1} . Choose any n -simplex $\alpha \cdot (c \setminus v) \in K$ which is disjoint from σ^{n-1} . We assert that $\alpha \cdot (c \setminus w) \in K$ for all $w \in c$: otherwise, c being a circuit, we will not be able to augment $c \setminus w$ by means of some subset of $\alpha \cdot (c \setminus v)$. And, all of these n -simplices $\alpha \cdot (c \setminus w)$, $w \in c$, being disjoint from σ^{n-1} , must have all but 2 of the vertices of $L_K \sigma^{n-1}$. This is only possible if, like v , all the vertices of c belong to $L_K \sigma^{n-1}$.

Two vertices are called *connected* in the matroid M_K if they belong to the same circuit of K . The transitivity property (2.3.4) tells us that this is an equivalence relation on V_K . The equivalence classes are called the *components* of M_K . From (2.4) we see that the components are at least as big as the codimension-one links $L_K \sigma^{n-1}$. In order to obtain an upper bound on the components of M_K , we will now verify that (2.2) is best possible.

(2.5) For all $\sigma^{n-1} \in K$, $\theta^n \in K$, the simplex $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n)$ does not lie in K .

Once again the proof will be by induction on the cardinality of $\sigma^{n-1} \cap \theta^n$.

If σ^{n-1} is disjoint from θ^n , then $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n)$ has $n+2$ elements, and so cannot be in K .

Now for the inductive step. If possible assume that $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n) \in K$ and choose any $v \in L_K \sigma^{n-1} \setminus \theta^n$ (such a v exists because $L_K \sigma^{n-1} \subseteq \theta^n$ would yield $L_K \sigma^{n-1} \in K$). *Case 1.* If the dimension of $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n)$ is less than n , then we can augment it by a vertex $w \neq v$ of the n -simplex $v \cdot \sigma^{n-1}$. So $w \in \sigma^{n-1} \cap \theta^n$. Now let $\varphi^n = u \cdot (\theta^n \setminus w)$ be an n -simplex of K disjoint from $w \cdot (\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n) \in K$. Since $u \notin L_K \sigma^{n-1}$ the inductive hypothesis now gives us the desired contradiction $w \cdot (\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n) = (\sigma^{n-1} \setminus \varphi^n) \cdot (L_K \sigma^{n-1} \setminus \varphi^n) \notin K$. *Case 2.* If the dimension of $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n)$ is n , then we use the base exchange axiom (2.3.1) to find a vertex $w \neq v$ in $v \cdot \sigma^{n-1}$ such that $w \cdot (\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus v) \in K$. Note that $w \in \sigma^{n-1} \cap \theta^n$. Since K_* is a pseudomanifold there exists a vertex $u \notin \theta^n$, $u \neq v$, such that $(\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus v) \cdot u \in K$. Also, because of the same reason, we can find an n -simplex $\varphi^n = z \cdot (\theta^n \setminus w)$, $z \neq u$, which is disjoint from $w \cdot (\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus v)$. But this is possible only if $z = v$: otherwise $(z \cdot (\theta^n \setminus w), (\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus v))$ would have valence 3. Since $\sigma^{n-1} \cap \varphi^n = (\sigma^{n-1} \cap \theta^n) \setminus w$, we can now use the inductive hypothesis to get the desired contradiction $(\sigma^{n-1} \setminus \varphi^n) \cdot (L_K \sigma^{n-1} \setminus \varphi^n) = w \cdot (\sigma^{n-1} \setminus \theta^n) \cdot (L_K \sigma^{n-1} \setminus \theta^n \setminus v) \notin K$.

The required upper bound follows as an easy corollary:

(2.6) If $v \in L_K \sigma^{n-1}$ then the (matroidal) component C_v of v is contained in $S_K \sigma^{n-1}$. In particular, if σ^{n-1} is incident to less than $n+3$ n -simplices, then C_v is a proper subset of V_K .

If some vertex w , not lying in the closed star of σ^{n-1} , were connected to v in M_K , then, by (2.4), we would have an $\xi^{n-1} \in K$ such that $v \cdot \xi^{n-1} \in K$ and $w \cdot \xi^{n-1} \in K$. Applying (2.5) to σ^{n-1} and $v \cdot \xi^{n-1}$ we see that $(\sigma^{n-1} \setminus \xi^{n-1}) \cdot (L_K \sigma^{n-1} \setminus \xi^{n-1} \setminus v) \notin K$, while applying (2.2) to σ^{n-1} , $w \cdot \xi^{n-1}$, and $v \in L_K \sigma^{n-1} \setminus w \cdot \xi^{n-1}$ we see that $(\sigma^{n-1} \setminus \xi^{n-1}) \cdot (L_K \sigma^{n-1} \setminus \xi^{n-1} \setminus v) \in K$. The second part follows by noting that, under the given hypothesis on σ^{n-1} , a vertex $w \notin S_K \sigma^{n-1}$ does indeed exist: otherwise any $(\sigma^{n-1}, \theta^n) \in K_*$ would have valence ≤ 1 .

We can now show that

(2.7) An n -dimensional simplicial complex K , $n \geq 0$, for which K_* is a $(2n+1)$ -pseudomanifold must be a join of some simplicial complexes of the type σ_s^{2s-1} , $s \geq 1$.

We will use induction on n .

For $n=0$ we note that each zero dimensional simplex of K_* has valence one less than the number of vertices in K . So σ_0^2 is the only 0-dimensional complex for which K_* is a 1-pseudomanifold.

If $n \geq 1$, and K has an $(n - 1)$ -simplex having maximum possible valence $n + 3$, then the result follows by (2.1). Otherwise, by (2.6), we can choose a vertex v of K whose matroidal component C_v is a proper subset of V_K . Let L^p (resp. N^q) be the full subcomplex of K spanned by the nonempty set of vertices which are in C_v (resp. not in C_v). We assert that $K = L \cdot N$. The inclusion $K \subseteq L \cdot N$ is trivial (and is valid for all disjoint partitions of V_K). To see $L \cdot N \subseteq K$ we note that if some $\alpha \in L, \beta \in N$, were such that $\alpha \cdot \beta \notin K$, then any circuit of K contained in $\alpha \cdot \beta$ would have a vertex of $\alpha \subseteq C_v$, and also a vertex of $\beta \subseteq V_K \setminus C_v$. This is not possible because C_v is the component of v .

The easily verified *join formula*

$$(L \cdot N)_* \cong L_* \cdot N_* \tag{1}$$

tells us that L_* (resp. N_*) is a $(2p + 1)$ -dimensional (resp. $(2q + 1)$ -dimensional) pseudomanifold. Since $p + q + 1 = n$ both p and q are less than n . So, by the inductive hypothesis, L and N , and thus their join $K = L \cdot N$, are of the required type.

§3. CRITICALITY

Before continuing the proof of Theorem (1.1) we recall some well known

Obstruction Theory (3.1). If a continuous map $f: K \rightarrow \mathbb{R}^{2n}$ obeys

$$f(\sigma^{n-1}) \cap f(\theta^n) = \phi \forall (\sigma^{n-1}, \theta^n) \in K_*, \tag{1}$$

then one can define a $(2n + 1)$ -dimensional integral Z_2 -cocycle ν_f of K_* by

$$\nu_f(\omega^n, \varphi^n) = \text{deg} [\partial(\omega^n, \varphi^n) \xrightarrow{f^{(2)}} (\mathbb{R}^{2n})_*]. \tag{2}$$

Here $f^{(2)} = f \cdot f$ denotes the Z_2 -map $K_* \rightarrow (\mathbb{R}^{2n})_*$ defined by $f^{(2)}(tx + uy) = tf(x) + uf(y)$, and $\partial(\omega^n, \varphi^n)$ denotes the boundary of the oriented $(2n + 1)$ -simplex (ω^n, φ^n) ; the degree is well defined because $(\mathbb{R}^{2n})_*$ (i.e., $\mathbb{R}^{2n} \cdot \mathbb{R}^{2n}$ minus points of the type $\frac{1}{2}x + \frac{1}{2}x$) has the oriented Z_2 -homotopy type of the antipodal $2n$ -sphere S^{2n} . From (2) we see that

$$f(\omega^n) \cap f(\varphi^n) = \phi \Rightarrow \nu_f(\omega^n, \varphi^n) = 0; \tag{3}$$

thus ν_f measures the extent to which f fails to separate disjoint n -simplices of K .

It is easy to see that any two continuous maps $f_0, f_1: K \rightarrow \mathbb{R}^{2n}$ obeying (1) can be joined by a homotopy $f_t, t \in [0, 1]$, obeying

$$f_t(\tau^p) \cap f_t(\xi^q) = \phi \forall (\tau^p, \xi^q) \in K_*, (p + q = 2n - 2) \forall t \in [0, 1]. \tag{4}$$

Hence we can define a $2n$ -dimensional integral Z_2 -cochain c_{f_t} of K_* by

$$c_{f_t}(\sigma^{n-1}, \theta^n) = \text{deg} [\partial((\sigma^{n-1}, \theta^n) \times [0, 1]) \xrightarrow{f_t^{(2)}} (\mathbb{R}^{2n})_*]. \tag{5}$$

This cochain satisfies the coboundary formula

$$\delta c_{f_t} = \nu_{f_t} - \nu_{f_0}, \tag{6}$$

which shows that the $(2n + 1)$ -dimensional integral Z_2 -cohomology class ν_K of K_* determined by ν_f is independent of f , and depends solely on K . (We remark that the mod 2 reduction of van Kampen's obstruction class ν_K is the $(2n + 1)$ th power of the Stiefel-Whitney class of the 2-fold cover $K_* \rightarrow K_*/Z_2$.) Also from (5) it follows that

$$f_t(\sigma^{n-1}) \cap f_t(\theta^n) = \phi \forall t \in [0, 1] \Rightarrow c_{f_t}(\sigma^{n-1}, \theta^n) = 0; \tag{7}$$

thus c_{f_t} measures the extent to which the homotopy f_t fails to obey (1) for $0 < t < 1$.

From the definition of ν_K just given it is easy to see that

$$\nu_K = 0 \text{ iff there exists a } Z_2\text{-map } K_* \rightarrow S^{2n}. \tag{8}$$

For $n \neq 2$ this homotopy theoretical condition is known to be equivalent to a topological one:

VAN KAMPEN-WU-SHAPIRO THEOREM (3.2). *An n -dimensional simplicial complex K , $n \geq 1$, $n \neq 2$, embeds in \mathbb{R}^{2n} iff $\nu_K = 0$.*

The original argument of van Kampen [15, corrected version], 1932, contained an unproved lemma, viz, the *p.l.* version of the (now) well known Whitney Trick [20]. The first complete proofs were given (independently) by Wu [21] and Shapiro [13].

(3.3) *Proof of (a) \Rightarrow (c).* We first show that K_* is homogenously $(2n + 1)$ -dimensional, i.e. that any $(\alpha, \beta) \in K_*$ is contained in a $(2n + 1)$ -simplex $(\sigma^n, \theta^n) \in K_*$: Being non-embeddable in \mathbb{R}^{2n} , K has certainly more than $n + 1$ (in fact more than $2n + 2$) vertices; so w.l.o.g. we can assume $\alpha \neq \beta$, $\beta \neq \phi$. Choose any $x \in \text{int } \alpha$, $y \in \text{int } \beta$, and let $f_{xy}: K \rightarrow \mathbb{R}^{2n}$ be continuous with no double points other than (x, y) contained in disjoint simplices of K . Thus f_{xy} , and so also any sufficiently near general position map $f: K \rightarrow \mathbb{R}^{2n}$, obeys $f(\sigma^n) \cap f(\theta^n) = \phi$ for all $(\sigma^n, \theta^n) \in K_*$ other than those for which $x \in \sigma^n$, $y \in \theta^n$, i.e. other than those for which $\alpha \subseteq \sigma^n$, $\beta \subseteq \theta^n$. Since $\nu_f \neq 0$ it follows from (3.1.3) that there must be a (σ^n, θ^n) of the last kind.

Now we show that K_* is a *pseudomanifold*, i.e. that any $(\alpha^{n-1}, \beta^n) \in K_*$ is incident to precisely two $(2n + 1)$ -simplices: Again, choose $x \in \text{int } \alpha^{n-1}$, $y \in \text{int } \beta^n$, $f_{xy}: K \rightarrow \mathbb{R}^{2n}$, and a sufficiently near general position map $f: K \rightarrow \mathbb{R}^{2n}$, so that $f(\sigma^n) \cap f(\theta^n) = \phi$ for all $(\sigma^n, \theta^n) \in K_*$ other than those for which $\sigma^n \supseteq \alpha^{n-1}$, $\theta^n = \beta^n$. If there were only one such σ^n , then (3.1.3) shows that ν_f is the coboundary of some multiple of the elementary Z_2 -cochain (α^{n-1}, β^n) . Since $\nu_K \neq 0$ it follows that there are at least two such σ^n 's, say $\sigma_1^n, \sigma_2^n, \dots, \sigma_r^n$, $r \geq 2$. We have to show now that $r = 2$. Choose any arc x_t , $0 \leq t \leq 1$, in $St_K \alpha^{n-1}$, from an $x_0 \in \text{int } \sigma_1^n$ to some $x_1 \in \text{int } \sigma_2^n$ via $x_t = x \in \text{int } \alpha^{n-1}$. The homotopy $f_t = f_{x_t y}: K \rightarrow \mathbb{R}^{2n}$ satisfies (3.1.4); besides, the two 'ends' f_0, f_1 , satisfy (3.1.1). Thus $\nu_{f_0}, \nu_{f_1}, c_{f_t}$, are well defined. Moreover, by using (3.1.3) and (3.1.7), we see that ν_{f_0} (resp. ν_{f_1}) is some multiple k_0 (resp. k_1) of the elementary Z_2 -cochain (σ_1^n, β^n) (resp. (σ_2^n, β^n)), while c_{f_t} is the some multiple k of the elementary Z_2 -cochain (α^{n-1}, β^n) . The coboundary formula (3.1.6) shows that this can happen only if $k = -k_0 = k_1$ and $r = 2$.

The remaining argument will be more geometrical than combinatorial or topological. We first need a straightforward generalization of a method employed by van Kampen [15], pp. 77-78 (to show that σ_n^{2n+2} and $\sigma_0^{2n} \dots \sigma_0^{2n}$ ($n + 1$ times) are non-embeddable in \mathbb{R}^{2n}). It depends on the following lemma whose proof is omitted.

(3.4). *Let $g: \sigma_n^{2n+2} \rightarrow \mathbb{R}^{2n}$ be linear with the images of $2n + 1$ of the vertices determining a $2n$ -simplex of \mathbb{R}^{2n} whose interior contains the images of the other 2 vertices. Then, there exists a unique pair of disjoint simplices (α, β) of σ_n^{2n+2} with $g(\text{int } \alpha) \cap g(\text{int } \beta) \neq \phi$.*

In fact if $g(v) = 1/2n + 1$, $\sum_0^{2n} g(u_i)$, $g(w) = \sum_0^{2n} c_i g(u_i)$, $0 < c_i < 1$, $c_0 + \dots + c_{2n} = 1$, and $c_0 \leq c_1 \leq \dots \leq c_{2n}$, then $\alpha = \{v\} \cup \{u_k: c_k > c_n\}$ and $\beta = \{w\} \cup \{u_k: c_k < c_n\}$, is the unique pair of disjoint open simplices intersecting under g .

(3.5) *Proof of (d) \Rightarrow (a).* Let us first consider the join-irreducible case $K = \sigma_n^{2n+2}$. For any given pair of points (x, y) contained in the interiors of disjoint open simplices (α, β) a map $f_{xy}: K \rightarrow \mathbb{R}^{2n}$ of the required kind can be constructed as follows. Choose $v \in \alpha$, $w \in \beta$. We imag

the other $2n + 1$ vertices to the vertices of a $2n$ -simplex $\subset \mathbb{R}^{2n}$, and then image v to their barycenter \bar{v} . On $K \setminus St_K w$, f_{xy} will coincide with the linear map thus determined. Let \bar{x} denote the image of x . The image \bar{y} of y will also be \bar{x} . If $y = w = \beta$, the linear map $K \rightarrow \mathbb{R}^{2n}$ thus determined is our f_{xy} . If $y \neq w$, then f_{xy} will be linear only on K_y , the simplicial complex obtained by deriving K at y . Note that now y is the interior point of a unique line segment zw with $z \in K \setminus St_K w$. Join its image \bar{z} to $\bar{y} = \bar{x}$ and extend to some \bar{w} lying in the interior of the $2n$ -simplex. We let \bar{w} be the image of w . Note that under this linear map $f_{xy}: K_y \rightarrow \mathbb{R}^{2n}$, the images of the simplices are the same as under the linear map $g: K \rightarrow \mathbb{R}^{2n}$ defined by the values of f_{xy} on the vertices of K . Applying (3.4) to g we see that (x, y) is the unique double point of f_{xy} which is contained in a disjoint pair of simplices of K . (See Fig. 1.)

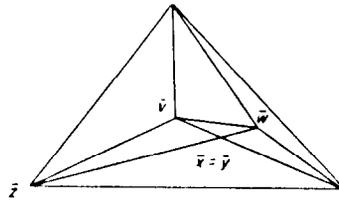


Fig. 1.

If further α, β are disjoint n -simplices (the subcase considered by van Kampen) then the cocycle ν_f is defined for $f = f_{xy}$, and is zero on all $(2n + 1)$ -simplices of K_* other than (α, β) , on which $\nu_f(\alpha, \beta) = \pm 1$. Thus the value of ν_f on an integral Z_2 -cycle formed by suitably orienting all the top-dimensional simplices of $(\sigma_n^{2n+2})_*$ equals ± 1 . This shows that $\nu_K \neq 0$, i.e. that there exists no Z_2 -map $K_* \rightarrow S^{2n}$. (Anticipating (d) \Rightarrow (b) — see (3.6) below — we see that van Kampen’s argument of 1932 also yields the Borsuk–Ulam Theorem of [1], 1933: “There exists no Z_2 -map from S^{2n+1} to S^{2n} .”) Thus K is non-embeddable in \mathbb{R}^{2n} .

We next show that f_{xy} can always be prolonged to a continuous family $f_{x'y'}$ of such maps defined for all (x', y') close enough to (x, y) . To do this let $\lambda = \bar{y}\bar{w}/y\bar{w}$ (if $y = w$ take $\lambda = 1$) and repeat the above construction with (x', y') taking care that $\bar{y}'\bar{w}'/y'\bar{w}' = \lambda$. For all (x', y') close enough to (x, y) , \bar{w}' will still be in the interior of the $2n$ -simplex, and so each of these maps $f_{x'y'}$ will also have a unique double point (x', y') which is contained in some disjoint pair of simplices of K .

Reducible case. Let $K^n = L^p \cdot N^q$, $n = p + q + 1$, where L^p (resp. N^q) is critically non-embeddable in \mathbb{R}^{2p} (resp. \mathbb{R}^{2q}). We assert that K^n is critically non-embeddable in \mathbb{R}^{2n} . We will only indicate how a map f_{xy} of the required kind can be defined when x (resp. y) is an interior point of a segment $x_L x_N$, $x_L \in L^p$, $x_N \in N^q$ (resp. $y_L y_N$, $y_L \in L^p$, $y_N \in N^q$). Note that x, y lie in disjoint simplices of K^n iff x_L, y_L and x_N, y_N lie in disjoint simplices of L and N respectively. But $\mathbb{R}^{2n} = \mathbb{R}^{2p+2q+2} = \mathbb{R}^{2p+1} \oplus \mathbb{R}^{2q+1}$. Let A^{2p} (resp. A^{2q}) denote a hyperplane in \mathbb{R}^{2p+1} (resp. \mathbb{R}^{2q+1}) at distance 1 from the origin. Let $f_{x_L y_L}: L^p \rightarrow A^{2p}$, $f_{x_N y_N}: N^q \rightarrow A^{2q}$ be 2 maps furnished by the criticality of L^p and N^q . Their join, perturbed slightly (see Fig. 2), will

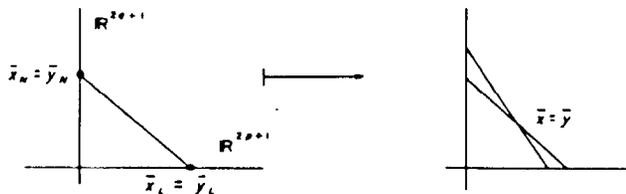


Fig. 2.

give an $f_{xy}: K \rightarrow \mathbb{R}^{2n}$ having just one double point (x, y) which is contained in some pair of disjoint simplices of K^n .

(3.6). *Proof of (d) \Rightarrow (b)* (Flores [2], 1933). For the join-irreducible case $K = \sigma_n^{2n+2}$, $n \geq 0$, this follows by verifying (cf. also Grünbaum [3, 4]) that the vertices v of a regular $(2n + 2)$ -simplex of \mathbb{R}^{2n+2} centred at the origin, and their antipodes \bar{v} , determine a convex $(2n + 2)$ -dimensional polytope whose boundary is combinatorially isomorphic to $(\sigma_n^{2n+2})_*$. (See Fig. 3 for the case $n = 0$.) The *reducible case* follows from this and the join formula (2.7.1). (Flores had used his lemma $K_* \cong S^{2n+1}$, and the recently established Borsuk–Ulam Theorem [1] to give an independent proof of the fact that there exists no Z_2 -map $K_* \rightarrow S^{2n}$, and so that K is non-embeddable in \mathbb{R}^{2n} .)

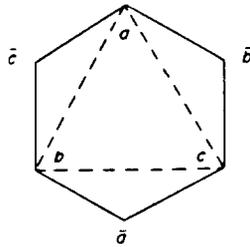


Fig. 3.

Since the implication (b) \Rightarrow (c) is trivial, this concludes the proof of Theorem(1.1)

(3.7) *Remarks.*

(1) The above theorem suggests that an n -dimensional simplicial complex K , $n \geq 1$, does not embed in \mathbb{R}^{2n} iff a Kuratowski n -complex “occurs” (in some sense which we hope to make precise elsewhere) in K .

However (and this must have been known to van Kampen) if K^n , $n \geq 3$, is *minimally non-embeddable in \mathbb{R}^{2n}* , and a p.l. space X^n is obtained from K^n by identifying two p.l. homeomorphic subpolyhedra, of codimension ≥ 2 , contained in the interiors of adjacent simplices of K , then X^n is also *minimally non-embeddable*. In fact, if there were a one-one continuous map $g: X^n \rightarrow \mathbb{R}^{2n}$, then, for any general position map $f: K \rightarrow \mathbb{R}^{2n}$ sufficiently close to the composition $K^n \xrightarrow{\text{quot.}} X^n \rightarrow \mathbb{R}^{2n}$, one would have $\nu_f = 0$, and so by (3.2) K^n would be embeddable in \mathbb{R}^{2n} . On the other hand all proper subspaces of X^n are obtainable from the embeddable proper subspaces of K^n by making codimension ≥ 2 identifications, and so, by Hilfsatz 6, p. 153, of [15], are also embeddable in \mathbb{R}^{2n} . (A different, but related, way of exhibiting an infinity of non-homeomorphic minimally non-embeddable spaces is given in Zaks [23].)

Thus, in higher dimensions, one cannot always expect the aforementioned “occurrence” to be as a subspace. We do not know whether it must be as a subspace *modulo* identifications of the above kind. Another related question: are the Kuratowski n -complexes the only ones which are minimally non-embeddable in \mathbb{R}^{2n} and for which each $(n - 1)$ -simplex has valence ≥ 3 ?

(2) In the definition of “critically non-embeddable” it is necessary to assume that (x, y) is in a pair of disjoint simplices: otherwise $\nu_f = 0$ for any g.p. map near to f_{xy} , and so, for $n \neq 2$,

K embeds in \mathbb{R}^{2n} . Note also that the argument given in (3.3) works with notions of "criticality" à priori weaker than that given in §1; however we do not know whether (a) \Rightarrow (c) holds without any local continuity condition whatsoever.

Let us say that a K^n , which does not embed in \mathbb{R}^{2n} , is *subcritically non-embeddable* if, for each (x, y) lying in a disjoint pair of simplices, there exists a continuous map $f_{xy}: K \rightarrow \mathbb{R}^{2n}$ under which K has a unique simple self-intersection at (x, y) . For $n \neq 2$, the deleted join K_* of such a K must be a minimal $(2n+1)$ -dimensional mod 2 cycle. The homogeneity of K_* can be checked as before. The remaining conclusion follows by noting that there is a nonzero $(2n+1)$ -dimensional minimal Z_2 -cycle z , over coefficients Z_{2^l} , l large, such that $v_K(z) = \pm 1 \pmod{2^l}$ (this follows because v_K is of order 2), and, since $v_K(z) = v_{f_{xy}}(z)$ for $(x, y) \in (\text{int } \sigma^n, \text{int } \theta^n)$, every $(2n+1)$ -oriented simplex (σ^n, θ^n) must occur in z with coefficients ± 1 .

Thus $(\sigma_0^2 \dots \sigma_0^2(n+1) \text{ times})$ is the only n -dimensional simplicial complex which is subcritically non-embeddable in \mathbb{R}^{2n} and for which each $(n-1)$ -simplex has valence 3. This follows because the mod 2 cycle K_* is clearly a pseudomanifold, and so we can use (c) \Rightarrow (d). Yet another corollary is that *there are only finitely many n -complexes K which are subcritically non-embeddable in \mathbb{R}^{2n} and for which the codimension one valences are all odd and less than a given number*. The finiteness can be checked by an easy combinatorial argument starting from the fact that, K_* being a mod 2 cycle, each θ^n must meet any given $\overline{\text{St}_K \sigma^{n-1}}$.

(3) *Bibliographical*. Kuratowski's theorem was anticipated by Pontrjagin—see note 5 of [6]—and discovered simultaneously also by Frink and P. A. Smith: see Whitney [18]. (The many known proofs of this theorem—see e.g. [14], and also [7] for a recent generalization to 2-manifolds other than \mathbb{R}^2 —can be interpreted as methods for eliminating double points in some low dimensional cases.) It still seems unknown whether $v_K = 0$ guarantees the existence of a *linear embedding* of K^n in \mathbb{R}^{2n} : see Grünbaum [4]. (Some—esp. linear—embedding problems seem to be easier if one works only with matroidal K 's: matroids were introduced in Whitney [19].) Lastly, note that Haefliger [5] discovered a generalized Whitney Trick which eliminates some higher dimensional non-isolated general position double points; this resulted in the generalization of (3.2) due to Weber [16].

(4) Classification theorems analogous to that of §2 are valid also for the higher order deleted joins of [10], e.g. one characterising those K 's for which the p th join configuration $K_*^{(p)}$ is a homogenous mod p cycle. These higher deleted functors measure (cf. Wu [22] and [11]) some obstructions to removing p -uple points, to embedding in dimensions lower than $2n$, etc. Thus these generalized classification theorems can also probably be put into a format analogous to that of Theorem (1.1). It still remains to work out the relationships between these developments, the known combinatorial interpretations of some characteristic classes, and the coloring results of [9], [10], etc. We hope to give a more leisurely and extensive account of the numerous combinatorial and topological applications of deleted functors in [12].

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