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A Generalized Kneser Conjecture

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We use the method of deleted joins to prove that if $N(j-1)-1 \geqslant M(p-1)+p(S-1)$, then any coloring of the S-subsets of an N-set by M colors must yield a p-tuple of S-subsets having the same color, *and* such that the intersection of any j of the sets is empty. © 1990 Academic Press, Inc.

1. INTRODUCTION

Kneser's Conjecture [6], 1955, was established by Lovász [7] in 1978. Likewise, a more general conjecture of Erdős [5], 1973, has now been proved by Alon, Frankl, and Lovász [1]. The object of this note is to show that such results can be obtained, in an almost canonical way, by using the deleted functor techniques of [8]. In fact we will use this method to prove the following result which is somewhat stronger. Here N, M, S, p, and j are positive integers, with $S \leqslant N$ and $j \leqslant p$, and a "j-wise disjoint" p-tuple is one in which the intersection of any j of the sets is empty.

THEOREM (1.1). *If* $N(j-1)-1 \geqslant M(p-1)+p(S-1)$, *then under any M-coloring of the S-subsets of an N-set, at least one j-wise disjoint p-tuple of S-subsets must be monochromatic.*

The aforementioned conjecture of Erdős is the "pairwise disjoint," i.e., $j=2$, case of this result, while the earlier Kneser Conjecture corresponds to the subcase $j=p=2$. Dirichlet's "pigeon-hole principle"—the common denominator in all Ramsey theoretic results—corresponds to the case $S=1$.

2. DELETED JOINS

We recall that a finite set is also called a *simplex* (of *dimension* one less than its cardinality) and a *simplicial complex* is any finite set K of simplices such that $\sigma \in K$, $\theta \subseteq \sigma$, implies $\theta \in K$. If the simplices of a simplicial complex

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A are disjoint from those of a simplicial complex B, then one defines the *join* $A \cdot B$ to be the simplicial complex consisting of all simplices $\alpha \cup \beta$, $\alpha \in A$, $\beta \in B$. The notation σ_t^i will denote the *t-skeleton on an i-simplex* σ^i, i.e., the simplicial complex consisting of all simplices $\theta^k \subseteq \sigma^i$, $k \leqslant t$.

DEFINITION (2.1). Let K be a simplicial complex and p a positive integer. Fix p copies, ${}^iK$, $1 \leqslant i \leqslant p$, of K, with mutually disjoint simplices, and denote the ith copy of a simplex $\sigma \in K$ by ${}^i\sigma \in {}^iK$. (For the empty simplex ϕ one has ${}^i\phi = \phi\ \forall i$.) *The simplex* ${}^1\sigma_1 \cup {}^2\sigma_2 \cup \cdots \cup {}^p\sigma_p$ *will be identified with the ordered p-tuple* $(\sigma_1, \sigma_2, ..., \sigma_p)$ *of simplices of* K. Thus the set of all such ordered p-tuples constitutes the p-fold join of K, $K^{(p)} = {}^1K \cdot {}^2K \cdot ... \cdot {}^pK$. $K_{(j)}^{(p)}$, the *j-wise disjoint pth join* of K, will consist of all those ordered p-tuples of simplices of K which are j-wise disjoint. So $K_{(j)}^{(p)}$, $j \leqslant p$, is an increasing sequence of subcomplexes of $K^{(p)}$. The simplicial complexes $K_{(2)}^{(p)}$ and $K_{(p)}^{(p)}$ will also be denoted by $K_\star^{(p)}$ and $K_*^{(p)}$, and called, respectively, the *pth join configuration* and the *pth deleted join* of K.

We now compute these deleted joins for a simplex:

$$(\sigma_i^i)_{(j)}^{(p)} \cong (\sigma_{j-2}^{p-1})^{(i+1)}. \tag{2.2}$$

For $i = 0$, this formula follows from the fact that any simplex of $(\sigma_0^0)_{(j)}^{(p)}$ is an ordered p-tuple having σ^0 in at most $j-1$ spots, and ϕ in all other spots. For $i > 0$ we note that σ_i^i is isomorphic to the $(i+1)$-fold join of σ_0^0. So (2.2) follows because, for any 2 complexes A and B with mutually disjoint simplices, one obviously has

$$(A \cdot B)_{(j)}^{(p)} \cong A_{(j)}^{(p)} \cdot B_{(j)}^{(p)}. \tag{2.3}$$

(2.4). Note that σ_{j-2}^{p-1} has the homotopy type of a bouquet of $(j-2)$-spheres. So its $(i+1)$-fold join, which by (2.2) is $(\sigma_i^i)_{(j)}^{(p)}$, *has the homotopy type of a bouquet of* $((i+1)(j-1)-1)$*-spheres*. In particular the pth deleted join $(\sigma_i^i)_*^{(p)}$ is homeomorphic to a $((i+1)(p-1)-1)$-sphere because it is the $(i+1)$-fold join of the $(p-2)$-sphere σ_{p-2}^{p-1}. On the other hand the pth join configuration $(\sigma_i^i)_\star^{(p)}$ is an i-dimensional space obtained by taking the $(i+1)$-fold join of p points.

(2.5). The group Σ_p of permutations of $\{1, 2, ..., p\}$, and therefore the subgroup $Z_p \subseteq \Sigma_p$ generated by the cyclic permutation $(2, 3, ..., p, 1)$, has an obvious action on the various subcomplexes of the p-fold join considered above: $\pi(\sigma_1, ..., \sigma_p) = (\sigma_{\pi(1)}, ..., \sigma_{\pi(p)})\ \forall \pi \in \Sigma_p$. Note that a simplex is Z_p-invariant iff it is of the type $(\sigma, \sigma, ..., \sigma)$. Since there are no such non-empty simplices in $K_{(j)}^{(p)}$, *the* Z_p*-action on* $K_{(j)}^{(p)}$ *is always fixed point free.* When p is prime Z_p is generated by any $\pi \in Z_p$, $\pi \neq \text{id}$, and so a simplex is Z_p-invariant iff it is π-invariant. Thus, *for p prime, the* Z_p*-action on* $K_{(j)}^{(p)}$ *is free.*

3. Colorings

We recall that the *barycentric subdivision* K', of a simplicial complex K, is the simplicial complex consisting of all chains (under $\subseteq$) of nonempty simplices of K. More generally, given any set U of simplices of K, $U' \subseteq K'$ will denote the subcomplex formed by all chains of simplices of U. Note that if $\varphi: K \to L$ is a monotone (under $\subseteq$) function from a simplicial complex K to a simplicial complex L, then it induces a simplicial map $\varphi': (\operatorname{supp} \varphi)' \to L'$. Here $\operatorname{supp} \varphi$, the *support of* φ, consists of all those simplices $\sigma \in K$ for which $\varphi(\sigma)$ is nonempty. We will also use the elementary fact that *if the nonempty simplices of K are partitioned into any two disjoint sets U and V, then K′ is isomorphic to a subcomplex of the join* $U' \cdot V'$. This follows because each chain of K' is determined by the two subchains formed by the members of U and V.

To prove (1.1) we'll use the following construction with $n = N-1$, $s = S-1$, and $m = M-1$.

Definition (3.1). Let f be a coloring of the s-faces of an n-simplex σ^n by the vertices of an m-simplex θ^m under which no j-wise disjoint p-tuple $(\xi_1^s, ..., \xi_p^s)$ of s-faces of σ^n is such that $f(\xi_1^s) = \cdots = f(\xi_p^s)$. Then we can define a monotone function $f_{(j)}^{(p)}: (\sigma_n^n)_{(j)}^{(p)} \to (\theta_m^m)_*^{(p)}$ by $f_{(j)}^{(p)}(\sigma_1, ..., \sigma_p) = (\bar{f}(\sigma_1), ..., \bar{f}(\sigma_p))$, where $\bar{f}(\sigma_i) \subseteq \theta_m^m$ denotes the set of colors assigned to s-faces of σ^n contained in σ_i. Note that $f_{(j)}^{(p)}$ is supported on the set of simplices other than those for which $\dim \sigma_i < s$ for all $1 \leqslant i \leqslant p$. These latter constitute the subcomplex $(\sigma_{s-1}^n)_{(j)}^{(p)}$ of $(\sigma_n^n)_{(j)}^{(p)}$. The join of $(f_{(j)}^{(p)})'$, and the identity map of $((\sigma_{s-1}^n)_{(j)}^{(p)})'$, yields a simplicial map from $X = ((\sigma_n^n)_{(j)}^{(p)})'$ to $Y = ((\sigma_{s-1}^n)_{(j)}^{(p)})' \cdot ((\theta_m^m)_*^{(p)})'$. This map will be denoted by F. Note that F commutes with the group actions considered in (2.5).

(3.2). *Proof of Theorem* (1.1). Using (2.4) we see that X is $((n+1)(j-1)-2)$-connected while Y has dimension $\leqslant (m+1)(p-1) - 1 + ps$. Thus, under the hypothesis of (1.1), one has connectivity $(X) \geqslant$ dimension (Y), and so, by the Borsuk–Ulam result of Dold [4, p. 68], one can have no map $F: X \to Y$ which commutes with some free actions of a non-trivial finite group on X and Y. Thus, by (2.5), there is no such coloring f when p is prime. We now mimic the easy proof of Proposition (2.3) of [1] to check that the validity of Theorem (1.1) for $p = p_1$ and $p = p_2$ implies its validity for $p = p_1 p_2$: Since $N(j-1) - 1 \geqslant M(p_1 p_2 - 1) + p_1 p_2 (S-1) = M(p_1 - 1) + p_1[M(p_2 - 1) + p_2(S-1)]$, the M-coloring of the N'-subsets $(N' = M(p_2 - 1) + p_2(S-1) + 1)$, which assigns to each N'-subset the color taken by all members of a chosen monochromatic pairwise disjoint p_2-tuple of S-subsets of this N'-subset, must itself yield a monochromatic j-wise disjoint p_1-tuple of N'-subsets. The chosen monochromatic p_2-tuples of

S-subsets within the members of this p_1-tuple furnish us with a $p_1 p_2$-tuple of S-subsets which is both j-wise disjoint and monochromatic. Q.E.D.

(3.3). The bound given by Theorem (1.1) is the *best possible*: If $N(j-1) \leqslant M(p-1)+p(S-1)=(M-1)(p-1)+(pS-1)$, then the M-coloring of the S-subsets of $\{1, 2, ..., N\}$ which assigns to each S-subset ξ having a vertex $\leqslant (M-1)((p-1)/(j-1))$ the color $]$(first vertex of $\xi)(j-1)/(p-1)[$ and to all other S-subsets the color M, cannot have a monochromatic j-wise disjoint p-tuple $(\xi_1, ..., \xi_p)$ of S-subsets. This follows because $\xi_1 \cup \cdots \cup \xi_p$ has cardinality $\geqslant Sp/(j-1)$ while the set of integers $\{t \mid (M-1)((p-1)/(j-1)) < t \leqslant N\}$ has cardinality $< Sp/(j-1)$.

(3.4). *Bibliographical Remarks*

(a) This work was done while establishing Conjecture 4 of Bogatyi [3]. Later on I came across [1] and realized that Bogatyi's conjecture coincided with that of Erdős.

(b) Sometimes it is more convenient to use *deleted products*, i.e., the subcomplexes of the cell complex ${}^1K \times {}^2K \times \cdots \times {}^pK$ defined analogously to the deleted joins of Definition (2.1). (This amounts to considering the subposets formed by ordered p-tuples of *nonempty* simplices of K.) Also note that analogous deleted functors can be defined in various other categories, e.g., those of Posets and Spaces. Deleted products and product configuration spaces have been used extensively by many authors notably in Embedding Theory; see, e.g., Wu [11].

(c) Results analogous to that of Dold [4] have been known for a long time and constitute a vast and rapidly expanding literature. All proofs of the Lovász–Kneser Theorem, or its generalizations or analogues, see, e.g., [1–3, 7, 8, 10], proceed via some such result. This is not surprising because Theorem (1.1) itself can be construed as a "Borsuk–Ulam result." We will make this remark more precise, and give some other applications of the constructions of this paper, elsewhere. A more leisurely and extensive account of the numerous combinatorial and topological applications of deleted functors will be given in [9].

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