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FORMAL THEORIES ARE ACYCLIC

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§ 1. Introduction

Corresponding to each formal theory or logic (i.e. a language together with its axioms and rules of inference) $\mathfrak{F}$ we will define a triangulated space (i.e. a geometrical simplicial complex) F. Indeed F will be the full simplicial complex whose vertices are the formulae of $\mathfrak{F}$ with two vertices being contiguous iff each is immediate from the other under the postulates of $\mathfrak{F}$. We will check that the notion of a proof in the formal theory $\mathfrak{F}$ corresponds to that of a path in the topological space F. Thus all the provable formulae can be characterised as the vertices of the component P of F containing the axioms; and, for any provable formula f, one can think of a proof as a path in P which joins it to some axiom g. With this picture in mind it is natural to enquire whether two such proofs of f are always homotopic to each other? An affirmative answer to this question follows if one can prove that the fundamental (or first homotopy) group $\pi_1(P)$ is trivial. We will prove the more general theorem that infact every component of F is acyclic in the sense that all its homotopy groups are trivial.

In 2.1 to 2.4 we will discuss the case when the formal theory is some first order logic. Analysing the proofs given here one sees that all the results extend to more general formal theories. In 2.5.2 we give an alternative construction of the topological space. And in 2.5.3 we point out that there are other natural, but more restrictive, notions of "homotopies of proof" under which one is led to the study of the homology of some subcomplexes (of this topologization) which are usually not acyclic.

The necessary background material in algebraic topology can be found in any textbook, e.g. in SPANIER [2].

§ 2. Topologization of formal theories

2.1. We consider the case when our formal theory $\mathfrak{F}$ is some *first order theory* or *logic*. The alphabet consists of

$$\sim \quad \wedge \quad \vee \quad \Rightarrow \quad \forall \quad \exists \quad) \quad (\quad , \quad c_i \quad v_j \quad p_{lk}$$

where the suffices i, j, k, l run over some sets of positive integers. Expressions $p_{lk}(x_1, x_2, \ldots, x_l)$, with each x_m, $1 \leqq m \leqq l$, equal to some constant c_i or else some variable v_j, are called *atomic formulae*; starting from these one builds up the set of all well formed formulae by employing the logical symbols in the usual way. A well formed formula is called a *formula* only if it has no free variables; the set of all formulae of $\mathfrak{F}$ will also be denoted by $\mathfrak{F}$. To describe the postulates of $\mathfrak{F}$ it will be convenient to let $\varphi(v)$ denote a well formed formula having v as its sole free variable; and if c is a constant, $\varphi(c)$ will denote the formula obtained from it by replacing all free occurences of v with c. By a set of *axioms* we mean any recursive subset $\mathfrak{A} \subset \mathfrak{F}$ containing all tautologous formulae as well as all formulae of the type $(\forall v)\, \varphi(v) \Rightarrow \varphi(c)$ and $\varphi(c) \Rightarrow (\exists v)\, \varphi(v)$. The *rules of inference* will be (i) modus ponens $\{f, f \Rightarrow g\} \to g$ and (ii) the generalization rules $f \Rightarrow \varphi(c) \to f \Rightarrow (\forall v)\, \varphi(v)$, $\varphi(c) \Rightarrow f \to (\exists v)\, \varphi(v) \Rightarrow f$ providing c does not occur in f nor in $\varphi(v)$.

2.2. A formula f will be called a *conjunction* of $f_1, f_2, \ldots, f_k$ if it is obtained from these formulae by employing the conjunction symbol $\wedge$ $k - 1$ times: e.g. $(f_1 \wedge ((f_2 \wedge f_3) \wedge f_4))$ is a conjunction of f_1, f_2, f_3, f_4. We will say that a formula f is *immediate* from a formula g if we can write f and g as conjunctions of $f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_l$ in such a way that each f_i is either (i) an axiom of $\mathfrak{F}$ or (ii) equal to some g_j or (iii) a direct consequence under generalization of an axiom or some g_j or (iv) a direct consequence under modus ponens of two formulae which are axioms or some g_j's. Further two formulae f and g will be said to be *contiguous* if each is immediate from the other.

We will now build a *triangulated topological space* F as follows. Each formula $f \in \mathfrak{F}$ is considered as a vertex; each pair $\{f_1, f_2\}$ of contiguous formulae of $\mathfrak{F}$ is considered as an edge with end points f_1 and f_2, ..., and more generally any set $\{f_1, f_2, \ldots, f_{n+1}\}$ of $n + 1$ pairwise contiguous formulae is considered as a closed n-dimensional simplex with vertices $f_1, f_2, \ldots, f_{n+1}$. The set theoretic union of these simplices is equipped with the weak topology and denoted by F.

Our object is to study some properties of this topological space in the following paragraphs.

2.3. We write $f \vdash g$ whenever there is a finite sequence $f = f_1, f_2, \ldots, f_n = g$ of formulae of $\mathfrak{F}$ with each f_i, $2 \leqq i \leqq n$, either an axiom or a consequence of some preceding formula(e) under the rules of inference: such a sequence is said to be a *proof* (of length n) of g from f. Formulae f and g are called *logically equivalent* if $f \vdash g$ and $g \vdash f$.

2.3.1. *Two formulae of $\mathfrak{F}$ are contained in the same component of the topological space F if and only if they are logically equivalent.*

2.3.2. Suppose that formulae f, g are contained in the component (maximal connected subset) C of F. We note that each point $f \in F$ has the path connected open neighbourhood $\mathrm{St}(f)$ given by taking the union of the interiors of all the simplices incident to f. Therefore a path component of C must be both open and closed and thus the whole of C. Hence one can find a continuous mapping Φ from the unit interval $[0, 1]$ into F such that $\Phi(0) = f$ and $\Phi(1) = g$. Replacing Φ by a suitable simplicial approximation, we can find a finite sequence $f = f_1, \ldots, f_n = g$ of formulae in C such that each f_i, $1 \leqq i \leqq n - 1$, is contiguous to f_{i+1}.

If $h \vdash j$ and k is immediate from j, then $h \vdash k$.

One can prove this in a routine manner by using definition 2.2 and the fact that the set of axioms contains all the tautologies. Using this lemma repeatedly on the above sequence we see that $f \vdash g$ and $g \vdash f$.

2.3.3. Conversely, since $f \vdash g$, we can find a proof $f = f_1, f_2, \ldots, f_n = g$ of g from f. Then the following sequence of conjunctions, where we have omitted writing the parantheses since their positions are of no consequence, has the property that each is contiguous to the preceding.

$$f, f_1 \wedge f_2, \ldots, f_1 \wedge f_2 \wedge \ldots \wedge f_{n-1} \wedge g, f_1 \wedge f_2 \wedge \ldots \wedge f_{n-2} \wedge g, \ldots, f_1 \wedge f_2 \wedge g, f \wedge g.$$

This shows that $f \wedge g$ lies in the same component as f. Likewise, from $g \vdash f$, we deduce that $g \wedge f$ lies in the same component as g. Since $f \wedge g$ and $g \wedge f$ are contiguous it follows that f and g are in the same component of F. This proves 2.3.1.

2.3.4. We will write $\vdash g$ if $f \vdash g$ for some axiom f; all such formulae g are called the *provable formulae* of $\mathfrak{F}$. (Conversely note that if f is an axiom, then $g \vdash f$ for all formulae g.) The subcomplex of F spanned by all the provable formulae will be denoted by P; 2.3.1 shows that P might also be defined as the component of F containing the axioms. The deduction lemma allows us to reformulate 2.3.1 as follows:

Formulae f, g of $\mathfrak{F}$ are in the same component of F iff $f \Leftrightarrow g$ is a provable formula.

One observes that the *negations* $\sim f$, $\sim g$ of two contiguous formulae f, g need not be contiguous; it is however true that *the subcomplex $\sim C$ spanned by the negations of all formulae of a component C of F is itself a component of F.* This corollary follows at once on observing that $f \Leftrightarrow g$ is provable iff $\sim f \Leftrightarrow \sim g$ is provable. We recall that the formal theory $\mathfrak{F}$ is called *consistent* if the set $\mathfrak{P}$ of provable formulae is disjoint from its negation $\sim \mathfrak{P}$. (For such theories it is easy to check that infact each component C is disjoint from its negation $\sim C$.) When the theory is not consistent every formula can be shown to be provable. Thus we see that *the topological space F is connected iff the formal theory $\mathfrak{F}$ is inconsistent.*

2.4. *Any continuous map from the n-sphere S^n, $n \geqq 1$, into F is homotopically trivial.*

2.4.1. For each finite set $\mathfrak{G} \subset \mathfrak{F}$ we define a subcomplex $G_c \subset F$ as follows: G_c is spanned by the set $\mathfrak{G}_c$ of formulae which are conjunctions of some formulae from $\mathfrak{G}$. We will prove that *any simplicial map Φ: $K \to G_c$, where K is some triangulation of S^n, $n \geqq 1$, is homotopically trivial.* This implies 2.4 because the simplicial approximation theorem tells us that one can find a triangulation K of S^n such that there is a simplicial map $K \to F$ in the same homotopy class as the given map $S^n \to F$.

2.4.2. A formula $f \in \mathfrak{G}_c$ will be said to have *weight* k if k is the least integer such that f is a conjunction (possibly with repetitions) of k distinct formulae from $\mathfrak{G}$; note that $1 \leqq k \leqq |\mathfrak{G}|$.

Case 1. If every formula $\Phi(v)$, v a vertex of K, is a conjunction (possibly with repetitions) of the same k distinct formulae from $\mathfrak{G}$ then obviously the map Φ: $K \to G_c$ is homotopic to a constant map.

Case 2. Otherwise we can choose a vertex v of K such that (i) $\Phi(v)$ has least weight k and (ii) there is a vertex w of K contiguous to v such that $\Phi(w)$ and $\Phi(v)$ are not conjunctions (possibly with repetitions) of the same k distinct formulae from $\mathfrak{G}$. The simplicial complex K is the union of the subcomplexes $\overline{\mathrm{St}(v)}$ and K-$\mathrm{St}(v)$; note that $\overline{\mathrm{St}(v)}$ is formed (see fig. 1) by coning its boundary $\mathrm{Lk}(v)$ ($= \overline{\mathrm{St}(v)} \cap (K\text{-}\mathrm{St}(v))$) over v. We construct a subdivision K' of K by retaining the subcomplex K-$\mathrm{St}(v)$ and by subdividing the subcomplex $\overline{\mathrm{St}(v)}$ into $(\overline{\mathrm{St}(v)})'$ as follows. For each vertex $a_i \in \mathrm{Lk}(v)$ we define $a_i' =$ midpoint of va_i (resp. $a_i' = a_i$) if $\Phi(a_i)$ is not (resp. is) the conjunction (possibly with repetitions) of the same k distinct formulae from $\mathfrak{G}$ as $\Phi(v)$. These vertices a_i' are now used to construct a complex $\mathrm{Lk}(v')$ which is isomorphic to $\mathrm{Lk}(v)$ under the correspondence $a_i' \to a_i$ (see fig. 2). We cone $\mathrm{Lk}(v')$ over v'. The remaining annular region between $\mathrm{Lk}(v)$ and $\mathrm{Lk}(v')$ is triangulated (in a standard way) as follows: We totally order the vertices of $\mathrm{Lk}(v)$ as $a_1, a_2, a_3, \ldots, a_N$. Then whenever $\{a_{i_0}, a_{i_1}, \ldots, a_{i_j}\}$, $i_0 < i_1 < \ldots < i_j$, is a simplex of $\mathrm{Lk}(v)$ and $0 \leqq p \leqq j$, we take $\{a_{i_0}, a_{i_1}, \ldots, a_{i_p}, a_{i_p}', \ldots, a_{i_j}'\}$ as a simplex for this annular region.

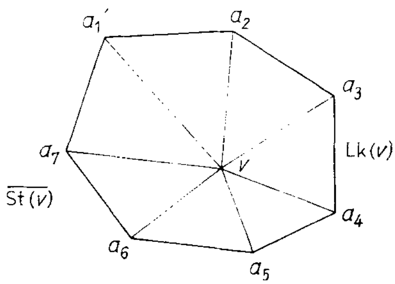

Fig. 1

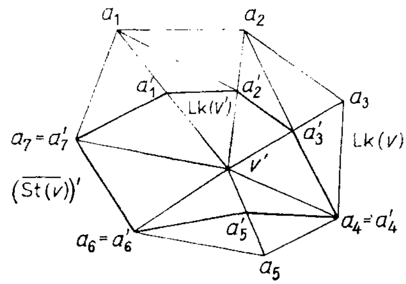

Fig. 2

We now define a new simplicial map Φ' of this triangulated n-sphere K' into F. Outside the region $\mathrm{St}(v)$ it will be same as Φ: for each vertex a_i of $\mathrm{Lk}(v)$ with $a_i' \neq a_i$ we let $\Phi'(a_i') = \Phi(a_i) \wedge \Phi(v)$ and lastly $\Phi'(v') = \Phi(v) \wedge \Phi(a_1) \wedge \ldots \wedge \Phi(a_N)$. Φ' is well defined because contiguous vertices of K' go to contiguous vertices of F.

(a) *The simplicial maps $\Phi, \Phi' : S^n \to G_c$ are homotopic.*

We identify the common boundary $\mathrm{Lk}(v)$ in the triangulations $\overline{\mathrm{St}(v)}$ and $(\overline{\mathrm{St}(v)})'$ of fig. 1 and fig. 2 and cone the resultant complex M over a new vertex x. One has the map $\Phi \cup \Phi' : M \to G_c$ which equals Φ on $\overline{\mathrm{St}(v)}$ and Φ' on $(\overline{\mathrm{St}(v)})'$. One can extend it to a simplicial map $\Psi : xM \to G_c$, e.g. by defining $\Psi(x) = \Phi(v)$. Since Φ, Φ' coincide outside the region $\mathrm{St}(v)$, this shows that $\Phi \simeq \Phi'$.

(b) *For any vertex a' of K' the formula $\Phi'(a')$ has weight $\geqq k$; further the number of vertices a' of K' for which $\Phi'(a')$ has weight k is one less than the number of vertices a of K for which $\Phi(a)$ has least weight k.*

This follows because (1) whenever $a_i' \neq a_i$ the weight of $\Phi'(a_i') = \Phi(a_i) \wedge \Phi(v)$ is strictly bigger than k and (2) the weight of $\Phi'(v') = \Phi(v) \wedge \Phi(a_1) \wedge \ldots \wedge \Phi(a_N)$ is strictly bigger than k.

If "Case 1" applies to Φ' we are through. Otherwise we can repeat the above process to get $\Phi'' \simeq \Phi$ etc. Since weight is bounded above by $|\mathfrak{G}|$ we see, from (b), that after a finite number of steps (say m) we will get a simplicial $\Phi^{(m)} \simeq \Phi$ such that $\Phi^{(m)}$ falls in "Case 1". Hence Φ is homotopically trivial.

2.4.3. *For every component C of F, the homotopy groups $\pi_n(C)$, $n \geqq 1$, are all trivial.*

This follows at once from 2.4 and the definition of these groups. Using the Hurewicz isomorphism theorem we now see that *the homology groups $H_n(F)$, $n \geqq 1$, are all trivial.*

Regarding $H_0(F)$ – it is the free Abelian group on the components of F – we have already seen in 2.3.4 that, for consistent $\mathfrak{F}$, it has rank $\geqq 2$. Infact it is fairly easy to see that *if a consistent first order logic $\mathfrak{F}$ is sufficiently rich, then F has $\aleph_0$ components.* Here by "sufficiently rich" we mean that the language should have enough predicates etc. to deduce GÖDEL's incompleteness theorem, viz., that the set T of true formulae (in any interpretation) is not recursively enumerable. (The subcomplex T of F spanned by T will be a disjoint union of components of F: further $T \cap (\sim T) = \emptyset$ and $T \cup (\sim T) = F$.) On the other hand one can deduce easily that the set $\mathfrak{C}$ of formulae lying in any component C of F is recursively enumerable. Since a finite union of r.e. sets is r.e. this rules out the possibility that F has a finite number of components.

2.5. By a *general theory* or *logic* $\mathfrak{F}$ we will understand a recursive set of formulae (also to be denoted by $\mathfrak{F}$) equipped with a finite number of recursive relations of which precisely one is singulary. This singulary relation $\mathfrak{A} \subseteqq \mathfrak{F}$ constitutes the *axioms* of our logic: the higher arity relations $\mathrm{R} \subseteqq \mathfrak{F} \times \ldots \times \mathfrak{F}$ are called the *rules of inference* of the logic (if $(f_1, f_2, \ldots, f_k) \in \mathrm{R}$ one says that f_k is a *direct consequence* of $\{f_1, f_2, \ldots, f_{k-1}\}$). The notion of *proof* and *provable formula* is defined just as before. (These definitions are almost the same as those given by DAVIS in [1], p. 117; as in that book we assume that there is a fixed countable alphabet, equipped with a Gödel numbering, out of which all the formulae are built.)

2.5.1. Let us suppose that $\mathfrak{F}$ is equipped with a binary operation $\wedge$ (or, alternatively that our alphabet has three symbols (,) and $\wedge$ such that if $f, g \in \mathfrak{F}$ then $(f \wedge g) \in \mathfrak{F}$). We can generalise all the definitions of 2.2, in the obvious way, to such logics $\mathfrak{F}$. A look at its proof shows that the acyclicity theorem 2.4 continues to hold even now. However one can prove 2.3.1 – which relates the topology of F to the notion of proof in $\mathfrak{F}$ – only under some additional conditions on $\mathfrak{F}$. It would suffice, e.g., to assume that $\mathfrak{F}$ is equipped with another binary operation $\Rightarrow$ such that (i) modus ponens is a rule of inference and (ii) all formulae of the type $f \wedge g \Rightarrow f$, $f \wedge g \Rightarrow g \wedge f$ and $f \Rightarrow (g \Rightarrow (f \wedge g))$ are axioms. This follows because (i) and (ii) allow us to deduce that formula k is immediate from formula j only if $j \vdash k$ and thus the lemma used in 2.3.2 holds.

2.5.2. There is an alternative way of constructing a suitable topological space which avoids imposing the extra conditions of 2.5.1 on $\mathfrak{F}$. Let $\tilde{\mathfrak{F}}$ be the set of all finite subsets of formulae and let $\mathfrak{F}$ be identified with the subset of $\tilde{\mathfrak{F}}$ consisting of singletons. We say that $\{f_1, f_2, \ldots, f_k\} \in \tilde{\mathfrak{F}}$ is *immediate* from $\{g_1, g_2, \ldots, g_l\} \in \tilde{\mathfrak{F}}$ if each f_i is either (i) an axiom or equal to a g_j or (ii) a direct consequence of some axioms and some g_j's under some rule of inference of $\mathfrak{F}$. Now (just as in 2.2) we define $\tilde{F}$ to be the full simplicial complex whose set of vertices is $\tilde{\mathfrak{F}}$ with two vertices being contiguous iff each is immediate from the other. The analogue of 2.3.1 can now be proved easily:

Two formulae of $\mathfrak{F}$ lie in the same component of $\tilde{F}$ iff they are logically equivalent in $\mathfrak{F}$.

(The component of $\tilde{F}$ containing the axioms will be denoted by $\tilde{P}$: a formula belongs to $\tilde{P}$ iff it is provable). Also we can prove, exactly as before, that *the topological space $\tilde{F}$ is acyclic.*

2.5.3. For each $n \geqq 1$ let F_n denote the subcomplex of $\tilde{F}$ spanned by $\mathfrak{F}_n \subseteqq \tilde{\mathfrak{F}}$ where $\mathfrak{F}_n$ consists of all subsets of $\leqq n$ formulae from $\mathfrak{F}$. The component of F_n containing the axioms will be denoted by P_n; clearly $P_n \subseteqq \tilde{P} \cap F_n$, $P_n \subseteqq P_{n+1}$ and $\bigcup P_n = \tilde{P}$. We will say that a provable formula is of *depth* $\leqq n$ if it belongs to P_n. $^{n \geqq 1}$

If a provable formula has a proof of length n, then it is of depth $\leqq n$.

This follows from an argument similar to that given in 2.3.3. Sometimes one has an upper bound for the depth: e.g., one verifies easily that *if the rules of inference of $\mathfrak{F}$ are all binary and symmetric, then any provable formula has depth* $\leqq 1$. (This remark applies to the logics associated to Thue or Post systems: see DAVIS [1], pp. 84, 117 for definitions.) For a provable formula f of depth $\leqq n$ one can restrict attention to those proofs which correspond to joining f to an axiom g by a path lying wholly within P_n: and, for two such proofs, it is natural to consider a more restrictive notion of homotopy in which one is not allowed to go outside the subcomplex P_n. Thus it is of interest to look at the homology and homotopy groups of P_n. Unlike $\tilde{P}$, which is always acyclic, these subcomplexes can have a very rich homology. Infact, *given any connected and full simplicial complex K having a countable number of vertices, one can find a logic $\mathfrak{F}$ for which $P_1 \cong K$.* (To see this let the alphabet be formed from the vertices $v_1, v_2, \ldots$ of K: let the set $\mathfrak{F}$ of formulae consist of all one letter words: let us have just one axiom v_1 and just one binary symmetric rule of inference R defined by $(v_i, v_j) \in \mathrm{R}$ iff $v_i v_j$ is an edge of K.) Since the barycentric derived of any simplicial complex is full, it follows that *we can find a logic $\mathfrak{F}$ whose P_1 is homeomorphic to any preassigned connected and countable simplicial complex.*

References

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