On Equivelar Maps

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§1. Introduction. This note is a by-product of [1], where I had discussed, for the benefit of its non-mathematical readers, some well-known examples of *finite uniform tilings* by p-gons, two on each edge, q around each vertex (the tiles were not required to be flat, nor the edges straight). It seemed to me that such *a finite uniform tiling should exist for any arbitrary* $p \geq 3$ *and* $q \geq 3$. However, I was unable to settle this point, nor find an answer later on in the literature. So I am presenting here what little I was able to do – mainly, the result that *this conjecture is true if* p *or* q *is even* – in the hope that this will arouse some interest and lead to its rapid resolution. Also, I'll switch here to a terminology that conforms better with previous work in this field.

Following Coxeter and Moser [2], page 20, a *map* shall mean a cell subdivision of a closed surface, that has finitely many zero-dimensional cells or *vertices,* to which finitely many one-dimensional cells or *edges* are attached by glueing their boundaries to pairs of (not necessarily distinct) vertices, and finally, finitely many two-dimensional cells or *faces* are attached, by glueing their circular boundaries to some closed edge paths (possibly with repeated vertices and edges) of this one-dimensional skeleton, so as to form the closed surface.

An *equivelar map of type* $\{p, q\}$ – or just 'a $\{p, q\}$' – is one in which each face has p edges (a repeating edge is counted twice) and each vertex is incident to q edges (an incident loop is counted twice). A $\{p, q\}$ on a non-orientable surface pulls back to a $\{p, q\}$ on its orientable 2-fold cover. So *the conjecture above is the same as saying that there exists a* $\{p, q\}$ *for any* $p \geq 3$ *and* $q \geq 3$. These two inequalities shall be in force from here onwards, and surfaces should be understood to be orientable, unless the contrary is indicated.

If a $\{p, q\}$ has N_0 vertices, N_1 edges and N_2 faces, then $qN_0 = 2N_1 = pN_2$, for example, $qN_0 = pN_2$ holds because each of the N_0 vertices is incident to q faces (each counted as many times as the vertex occurs in it) and each of the N_2 faces has p vertices (each counted as many times as it occurs in the face). So $N_0/2p = N_1/pq = N_2/2q$ and the Euler characteristic $2 - 2g$ of the surface equals this ratio times $2p - pq + 2q = 4 - (p - 2)(q - 2)$. Using this we'll show next that *any surface admits only finitely many types of equivelar maps.* Note that if the Euler characteristic is nonzero, the numbers N_i are fixed by the type $\{p, q\}$, so it follows that, excepting the 2-torus, any (orientable) surface admits only finitely many (combinatorial isomorphism classes of) equivelar maps.

The 2-sphere can admit only $\{p, q\} = \{3, 3\}, \{3, 4\}, \{4, 3\}, \{3, 5\}$ or $\{5, 3\}$ because $2p - pq + 2q > 0$ only for these $p, q \geq 3$, and conversely, it has essentially been known ever since Plato and Euclid, that it admits a unique equivelar map of each of these five types, viz., the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron. Again, $2p - pq + 2q = 0$ iff $\{p, q\} = \{3, 6\}, \{4, 4\}$ or $\{6, 3\}$, and conversely, it has been known since very long that the 2-torus admits infinitely many equivelar maps of each of these three types. For all other types, using $N_0 \geq 1$ and $p \geq 3$ we get $2 - 2g = \frac{N_0}{2p}[2p - pq + 2q] \leq 1 - \frac{q}{2} + \frac{q}{p} \leq 1 - \frac{q}{6}$, and similarly, using $N_2 \geq 1$ and $q \geq 3$, that $2 - 2g \leq 1 - \frac{p}{6}$, which show that we must have $p, q \leq 12g - 6$.

These general bounds on p or q are the best possible, for example, in § 6 there is a $\{18, 3\}$ of genus 2 with just one face. Unlike this, most of the $\{p, q\}$'s that we shall construct below are *non-degenerate,* that is, their edges and faces have distinct vertices. For these one has $N_0 > p$, so $2 - 2g < \frac{1}{2}[4 - (p-2)(q-2)]$, therefore, *the possible types of a non-degenerate equivelar map on a surface of genus* $g \geq 2$ *are the finitely many integral points lying in the shaded area of* Figure 0 *between the hyperbolas* $(p-2)(q-2) = 4$ *and* $(p-2)(q-2) = 4g$. The three toral types are on the hyperbola $(p-2)(q-2) = 4$, while the five below it live on the 2-sphere.

Figure 0

From Coxeter and Moser [2], pages 101-102, one learns that genus 2 equivelar maps were discussed in a 1922 paper of Errera, and much more fully, by Threlfall in his 1933 thesis; only, these authors had called them 'regular'. However, following some 1927 papers of Brahana, and then [2] itself, it is now usual to call a $\{p, q\}$ a *regular map* only if all its local rotations – that is, of any face, or around any vertex – can be extended to global combinatorial automorphisms. So it would have been apt to dub the original and much weaker notion a *locally regular map,* however we have gone for the shorter 'equivelar map' of McMullen and Schulte [3], page 20. Actually, the adjective 'equivelar' had been coined already in McMullen, Schulz and Wills [4], but in that paper, and its sequels [5] and [6], it was applied only to cell subdivisions of surfaces that can be *linearly embedded* in Euclidean 3-dimensional space with all faces convex polygons, a much stronger requirement than merely demanding that the map be non-degenerate. For example, a $\{p, 3\}$ with $p > 5$ cannot be linearly embedded in Euclidean n-dimensional space for any $n \geq 3$. The 'doubling construction' of § 3 below was used, in this geometric context, in [5] to iteratively construct some $\{3, q\}$'s and $\{4, q\}$'s linearly embeddable in 3-space.

In our purely topological context, doubling is more flexible because – as we show in § 3 and § 4 respectively – many, but unfortunately not all, of the 'duals' of the aforementioned iterated doubles also obey the 'strong disjoint covering property' of § 2. This allows us to employ a far-reaching generalization of the doubling construction in § 5 to construct *examples of non-degenerate equivelar maps of all types* $\{p, q\}$ *with* p *or* q *even.* Some more examples, constructions, and bibliographical remarks are given in the concluding § 6.

§2. Preliminaries. The Poincaré *dual* of a $\{p, q\}$ is the equivelar map of type $\{q, p\}$ of the same surface constructed as follows: in each face is chosen a barycenter or dual vertex, then a dual edge laid across each edge so as to join the dual vertices in the two incident faces. So each vertex is contained in one and only one dual face, namely that whose dual vertices and dual edges are those of the vertex's incident faces and edges. Thus if $\{p, q\}$ has N_0 vertices, N_1 edges and N_2 faces, then its Poincaré dual $\{q, p\}$ has N_2 vertices, N_1 edges and N_0 faces.

A map has the *disjoint covering property* or *d.c.p.* if its vertex set can be partitioned off into vertex sets of pairwise disjoint faces; and it has the *strong disjoint covering property* or *s.d.c.p.* if from the remaining faces a second pairwise disjoint set of faces covering all the vertices can also be chosen.

An obvious necessary condition for a non-degenerate $\{p, q\}$ to have the d.c.p. is that N_0/p be a whole number, but $N_0/p = N_2/q$, so it is also the necessary condition for its non-degenerate dual $\{q, p\}$ to have the d.c.p. The tetrahedron $\{3, 3\}$, and the 7-vertex triangulation of the torus, a $\{3, 6\}$, are examples not satisfying this necessary condition. This condition is far from being sufficient. In fact we'll check in §4 that, for p odd, a $\{p, 3\}$ cannot have the d.c.p., but $pN_2 = 3N_0$ shows that if p does not contain the prime 3 then p does divide N_0. The octahedron $\{3, 4\}$, and its dual, the cube $\{4, 3\}$, are examples having the s.d.c.p., because vertices not incident to any face form the vertex set of a disjoint face. Removing a pair of disjoint faces gives us the 'tubes' of Figure 1 whose vertices are covered by the pairs of similarly shaded disjoint faces.

Figure 1a Figure 1b

Again, although the dodecahedron $\{5, 3\}$ does not have the d.c.p., its dual, the icosahedron $\{3, 5\}$, has the d.c.p., because the four disjoint shaded triangles of Figure 2 cover its 12 vertices, and rotating these triangles around the vertical N-S axis shows further that it even has the s.d.c.p.

Figure 2

Though we don't have an explicit example, it seems likely that there are non-degenerate $\{p, q\}$'s that satisfy the d.c.p., but not the s.d.c.p.

§3. Doubling construction. Starting with the octahedron $\{3, 4\}$, respectively the icosahedron $\{3, 5\}$, one can inductively construct a non-degenerate $\{3, s\}$ satisfying the s.d.c.p for any even, respectively any odd s, by the following *doubling* construction, which has been used before in [5].

We delete from $\{3, s\}$ a set of disjoint faces covering all its vertices to obtain K. We then join each bounding triangle 123 of K to the corresponding bounding triangle $1'2'3'$ of a disjoint copy K' by a tube as in Figure 1a. This gives us a $\{3, s + 2\}$ satisfying the s.d.c.p., because similarly shaded triangles, two from each joining tube, cover all its vertices.

An analogous inductive doubling construction, starting from the cube $\{4, 3\}$ and using the tubes of Figure 1b also gives us, for any s, a non-degenerate $\{4, s\}$ obeying the s.d.c.p.

Figure 3a Figure 3b

We have redrawn Figure 1b as Figure 3b above, in which the dotted lines and white dots depict the (portions of the) edges and vertices of the dual $\{s+1, 4\}$ which lie in this tube. Note that we had chosen the vertices opposite $1, 2, 3$ and 4 to be $3', 4', 1'$ and $2'$ respectively. This twist will now be exploited to show that *the duals* $\{s, 4\}$ *also satisfy the s.d.c.p.*

Since this is so for the initial $\{3, 4\}$, it suffices to show that if $\{s, 4\}$ has the s.d.c.p., then $\{s+1, 4\}$ has the s.d.c.p. Consider any pairwise disjoint set of faces of $\{s, 4\}$ covering its vertices. One and only one of these s-gons has, as a vertex, the barycenter of the deleted square face 1234 of $\{4, s\}$. This s-gon is the dual face of one of the four vertices $1, 2, 3$ or 4, say of 3. The doubling construction changes this s-gon into an $(s+1)$-gon of $\{s+1, 4\}$, whose portion within the tube is shown shaded in Figure 3b, while the portion off the tube coincides with that of the original s-gon. Also shown shaded in Figure 3b is the portion of the $(s+1)$-gon contributed similarly by the disjoint copy. We note that these portions are disjoint and cover all the white dots which are in this tube. Thus the original pairwise disjoint covering set of faces of $\{s, 4\}$ has become a pairwise disjoint covering set of faces of $\{s+1, 4\}$, which proves the implication.

A similar argument, using Figure 3a this time, shows that the duals $\{s, 3\}$ of all the $\{3, s\}$, s even, which were obtained above by repeatedly doubling the octahedron, also satisfy the s.d.c.p.

§4. A remark. *No* $\{s, 3\}$ *with* s *odd can have the d.c.p.* Suppose one can find a set V of vertices of $\{3, s\}$ whose dual faces are disjoint and cover the dual vertices. Then each triangular face of $\{3, s\}$ is in the star of one and only one vertex from V. So, if v is in V, then any vertex u in the link of v cannot be in V. Let $v = v_1, v_2, \ldots, v_s$ be the vertices in the link of u in cyclic order. The vertex v_3 must be in V, otherwise the shaded triangle uv_2v_3 of Figure 4 would not be in the star of any vertex of V. Likewise $v_5, v_7, \ldots$ are in V, which rules out that s is odd, because v_s is in the link of v, so not in V.

Figure 4

For s even, the above argument shows that V is determined by any of its members v. In other words, the vertices of an $\{s, 3\}$ with s even can be covered by at most one set of disjoint s-gons which contains a specified s-gon.

§5. A generalization of doubling. We shall now show that, *if there exists a non-degenerate* $\{p, q\}$ *satisfying the d.c.p., then there also exists a non-degenerate* $\{p, q+2\}$ *satisfying the d.c.p.* For this purpose we generalize the doubling construction as follows.

Suppose that the vertices of the $\{p, 4\}$ satisfying the s.d.c.p., that we had constructed above in §3, can be covered by the disjoint faces $G_1, \ldots, G_t$. We delete these to obtain T, which shall serve as our 'generalized tube'. Note that T

is a surface with t disjoint p-gonal boundaries $\partial G_1, \ldots, \partial G_t$. We'll need disjoint copies T^j of T, the copy of ∂G_i in T^j shall be denoted by $(\partial G_i)^j$.

Choose next disjoint faces $F_1, \ldots, F_r$ of the given $\{p, q\}$ covering its vertices. Deleting these, one obtains K, a surface with disjoint p-gonal boundaries $\partial F_1, \ldots \partial F_r$. We'll be using t disjoint copies of K, the ith copy shall be denoted by K^i, with $(\partial F_j)^i$ denoting the copy of ∂F_j in it.

We now attach to the union of these t copies $K^1, \ldots, K^t$ of K, r disjoint copies $T^1, \ldots, T^r$ of the generalized tube T, by identifying the p-gonal boundary $(\partial F_j)^i$ of K^i with the p-gonal boundary $(\partial G_i)^j$ of T^j. This gives us a closed surface subdivided into p-gonal faces, with $q+2$ edges incident to each vertex. Furthermore, this non-degenerate $\{p, q+2\}$ satisfies the d.c.p., because t disjoint faces can be chosen in each T^j which cover the vertices lying in T^j, and taken together all these tr disjoint faces cover all the vertices of $\{p, q+2\}$.

Applying this construction to the $\{p, 4\}$ of §3 gives us a $\{p, 6\}$ with d.c.p., then applying the construction again to this we obtain a $\{p, 8\}$ with d.c.p., etc. Thus *one has an equivelar map* $\{p, q\}$ *for any* $p \geq 3$ *and any* $q > 3$ *even,* or, since we can always pass to the dual, for any $p > 3$ even and any $q \geq 3$. Alternatively, from §3 we have, for p even, a $\{p, 3\}$ with d.c.p. Applying the construction to it, one gets a $\{p, 5\}$ with d.c.p., then a $\{p, 7\}$, etc. Or, yet again, there is also a parallel construction, using this time as 'generalized tube' $\{p, 3\}$, p even, minus a covering set of disjoint p-gonal faces, which, applied repeatedly to the $\{p, 4\}$ of §3, also gives all $\{p, q\}$ with p even.

§6. Concluding remarks Though we don't have (p, q)'s for arbitrary pairs of odd numbers p and q bigger than 3, we can make *all* (p, p)*'s with* p *odd* as follows.

The $(5, 5)$ is in fact known since very long. According to many (see, e.g., [12], p.5) the marble inlay work – its photograph 5a is from [14] – in the floor of the Basilica of St. Mark's in Venice is by Paolo Uccello and dates from 1420. It depicts the "small stellated dodecahedron," the regular but non-convex polyhedron with self-intersections – see Coxeter [13], Chapter VI, for this and other Kepler-Poinsot polyhedra – which is obtained by extending the facets of the dodecahedron so that pentagonal pyramids get erected over each facet. The 12 pyramidal peaks are its vertices, the 30 extensions of the dodecahedral edges joining these peaks are its edges, while the 12 planar pentagrams extending the dodecahedral facets are considered to be its faces. Thus it is indeed a $(5, 5)$; moreover, it has genus four, because $2 - 2.4 = 12 - 30 + 12$. Thus, somewhat magically, just by extending the facets of the dodecahedron, we have obtained a closed surface quite different from the original sphere!

Figure 5a

Figure 5b

The same combinatorial and topological end can be achieved – see Figure 5b – by declaring a black dot in each face of $(5, 3)$ as a new vertex, and the extension of each edge of $(5, 3)$ to the black dots in the two faces, not incident to the edge, to which its two ends are incident, as a new edge. This makes sense, much more generally, for any simple polygonally subdivided closed surface, i.e., one which occurs as the Poincaré dual P of a triangulated surface T. We shall call the resulting polygonal complex U the *Uccello (surface)* of T or P.

Figure 6a Figure 6b

A face of P which is p-gonal with p odd gives rise – Figure 6a – to just one compatibly oriented face of U, also p-gonal; but if p is even, it gives rise – Figure 6b – to *two* compatibly oriented $p/2$-gonal faces of U. The vertex of U in this face of P is however incident to p edges of U in both cases. Also, the edge of U extending a given edge of P, occurs in one and only one of the possibly two faces of U associated to either of the two incident faces of P, and in no other face of U. Thus U is, like the original T, an orientable closed surface. Moreover, if the connected T has only vertices of odd valence, then U is connected. However, its genus is usually quite different, for *the Euler characteristic of* U in this case is $2N_0 - N_1$, where N_0, N_1 and N_2 denote the number of vertices, edges and triangles of T. When T has also some, say e, vertices of even valence, then the Euler characteristic of U is $2N_0 - N_1 + e$, because the cells of P dual to these vertices contribute not one, but two cells of U. For the same reason, in this case, U can be disconnected. For example, the Uccello surface of the 9-vertex toral $(3,6)$ shown in Figure 7a is the disjoint union of 3 isomorphic 3-vertex toral $(3,6)$'s, one of which is shown in Figure 7b.

Figure 7a Figure 7b

For p odd, the Uccello surfaces of the $(3,p)$'s of §3 are connected, and give the required examples of equivelar maps of type (p,p). However, *these* (p,p)*'s, p odd, do not satisfy the d.c.p.,* so we cannot use the construction of §4 to now make a $(p,p+2)$, etc., from them. In fact, for any T having only vertices of odd valence, it is true that U cannot satisfy the d.c.p. This follows because the vertex set of a cell of U – for example, $\{1,2,3,4,5\}$ of Figure 5b, or $\{1,2,3,4,5,6,7\}$ of Figure 6a – coincides with the vertex set of the link of a vertex $0 \in T$, and any cell of U having 0 as a vertex intersects the given cell.

It is worth noting also that, *there is a triangulation of the* 2*-sphere whose Uccello surface has any pre-assigned genus.* Starting with the tetrahedron $T_0 = A_0B_0C_0D_0$, which coincides with its Uccello, let T_s, $s \geq 0$, denote the triangulation of the 2-sphere with a prescibed triangle $A_sB_sC_s$, which is to be subdivided as in Figure 8 to make T_{s+1}. Then the Uccello of T_s is connected and has genus $2s$. This follows because the subdivision keeps the valence of the old vertices odd, and introduces four new odd valence vertices and 12 new edges; so the Euler characteristic of the Uccello changes by $2.4 - 12 = -4$ at each step. Consider next the triangulation R_s obtained by deriving the triangle $A_sB_sC_s$ of $T_s, s \geq 1$ at a new vertex. The Euler characteristic of its Uccello differs from that of T_s by $2.1 - 3 + 3 = 2$ because it has three even valence vertices, A_s, B_s, C_s. Nevertheless, despite these three 'bad' vertices, it can be checked that this Uccello is also connected, so its genus is $2s - 1$.

Figure 8

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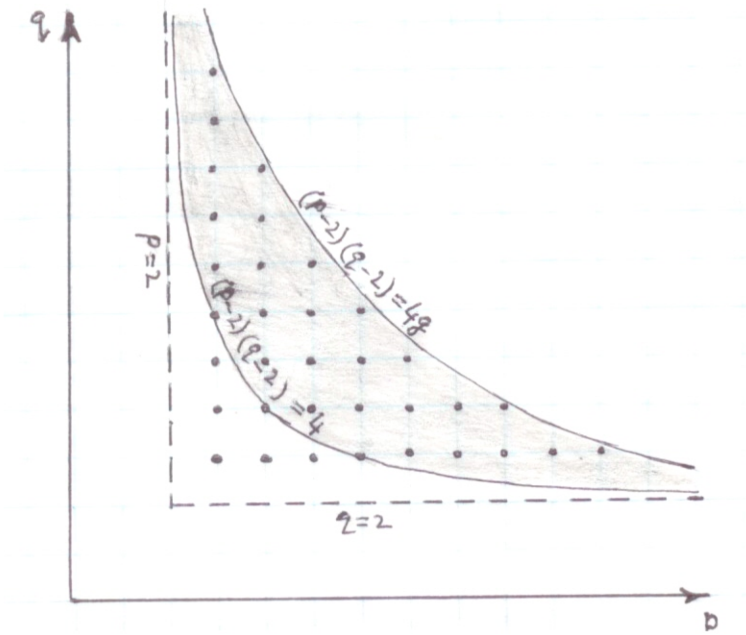

Figure 0

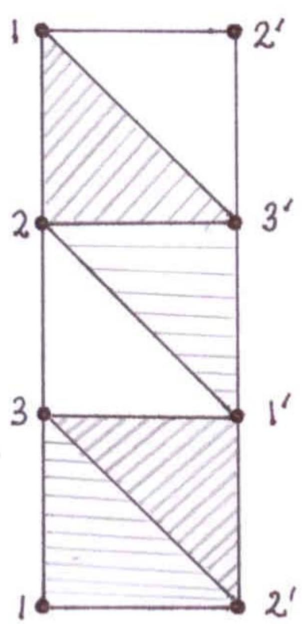

Figure 1a

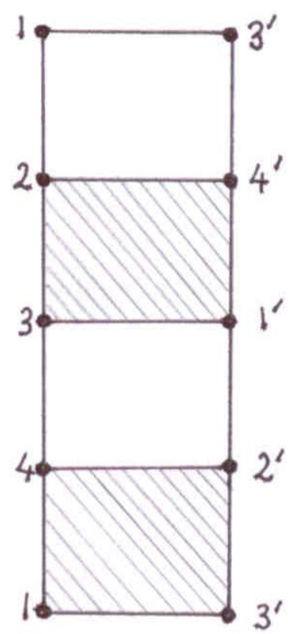

Figure 1b

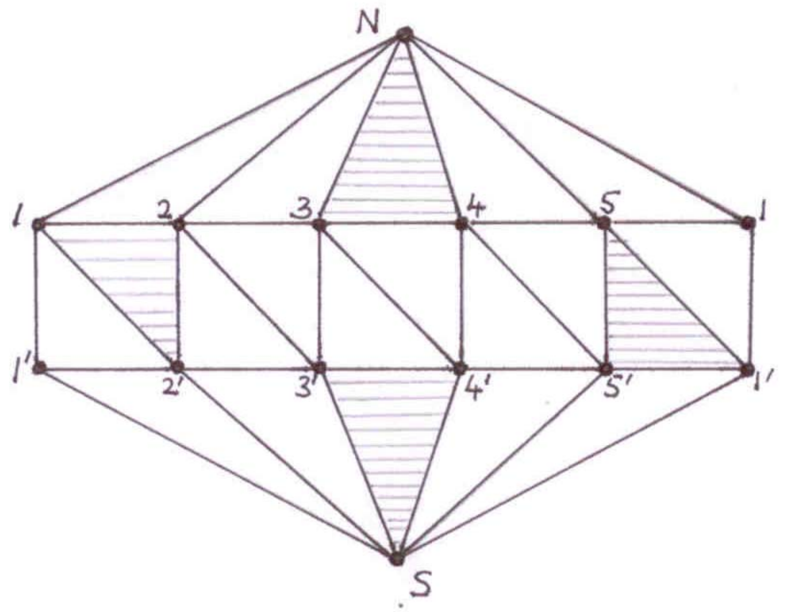

Figure 2

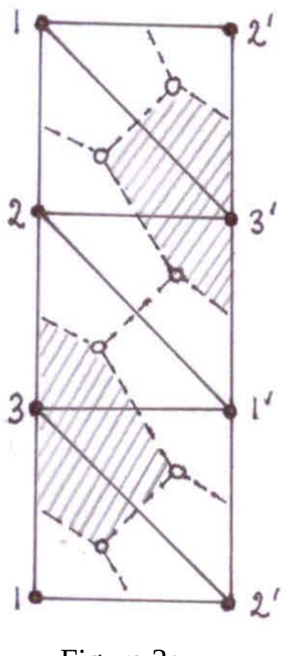

Figure 3a

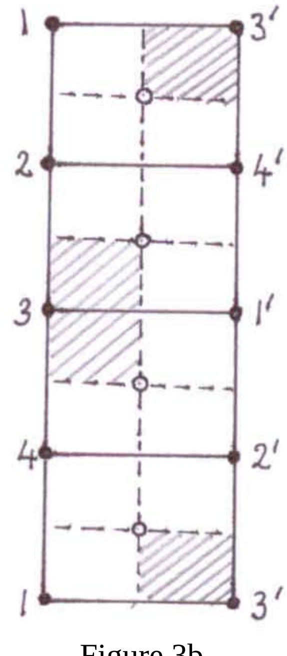

Figure 3b

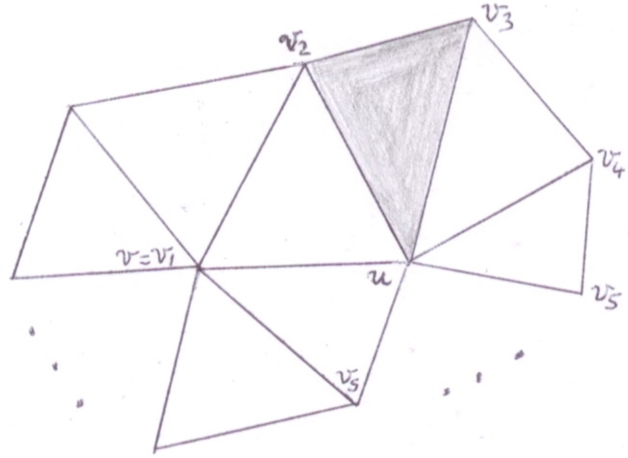

Figure 4

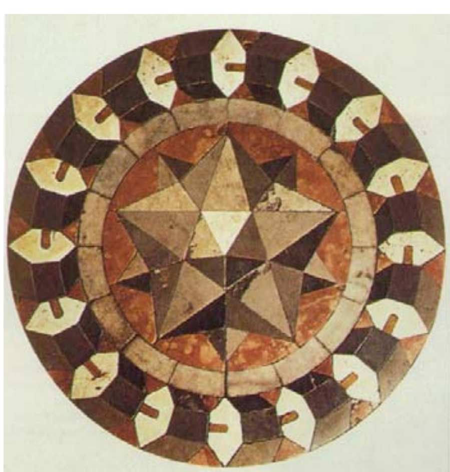

Figure 5a

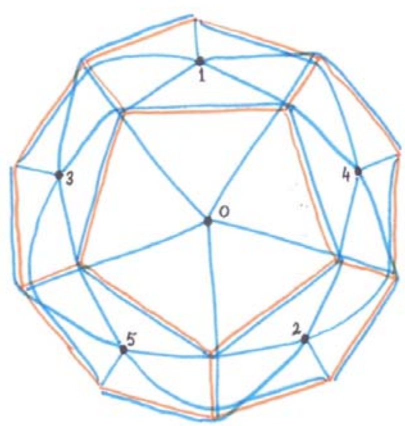

Figure 5b

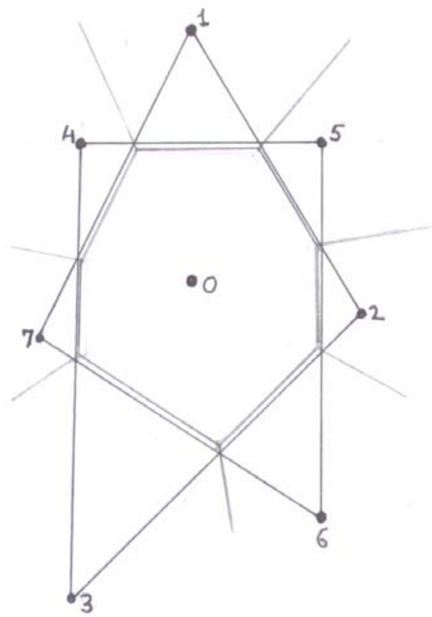

Figure 6a

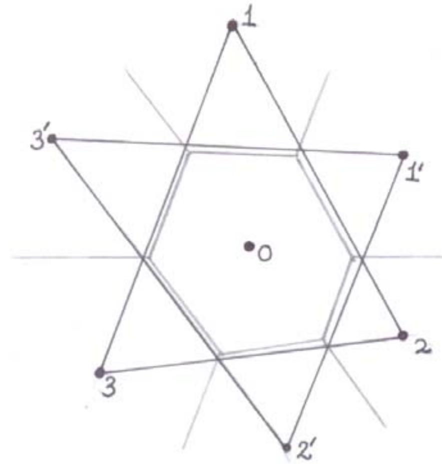

Figure 6b

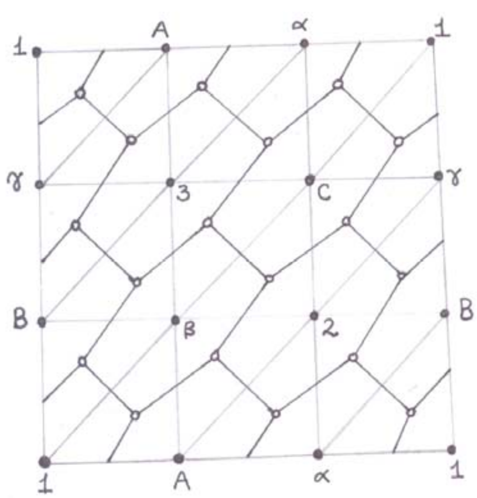

Figure 7a

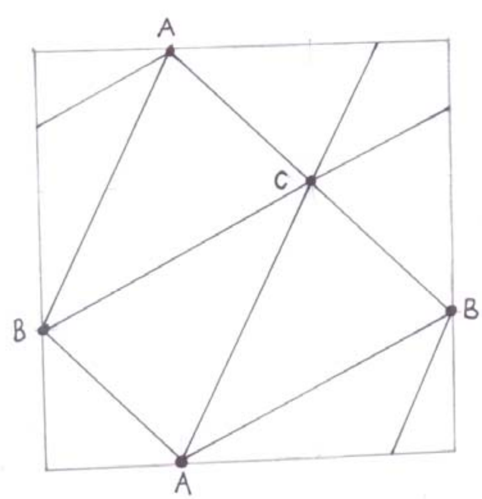

Figure 7b

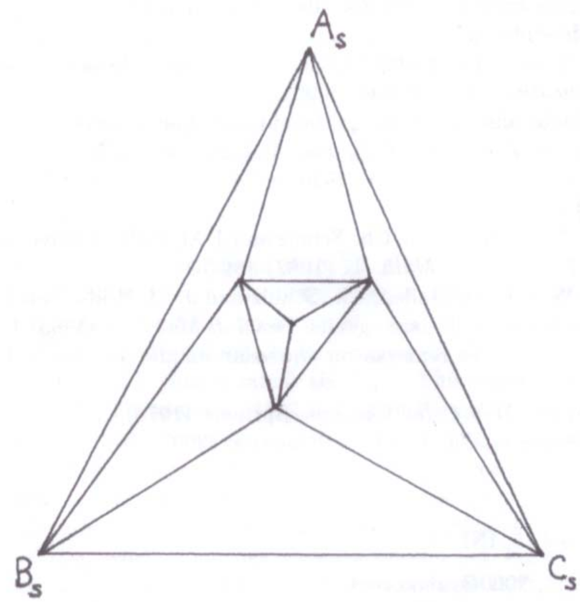

Figure 8