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# ADDENDUM TO MY PAPER "ON COLORING MANIFOLDS"

BY

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An important paper by Grünbaum [1], which had escaped my attention until now, contains the following theorem: If $m \geq 2$ then one can assign $6(m-1)$ colors to the $(m-2)$-simplices of any simplicial complex imbedded in $\mathbb{R}^m$ in such a way that any two $(m-2)$-simplices incident to the same $(m-1)$-simplex have different colors. A fortiori, this implies the finiteness of the numbers $\mathrm{ch}_{m-2}(S^m)$ of [2].

It is easily seen that Theorems 1 and 2 of [2] are equivalent to the following.

THEOREM A. *If $X$ is any closed $m$-dimensional pseudomanifold ($m \geq 2$), then*

$$\mathrm{ch}_{m-2}(X) \leq \left\{\frac{m(m+1)}{m-1}[1 + b_{m-1}(X; \mathbb{Z}_2)]\right\}.$$

*Further if $K$ is any subcomplex of a triangulation of $X$ and contains at least one $(m-2)$-simplex, then*

$$\frac{m-1}{m+1}\,\alpha_{m-1}(K) \leq \alpha_{m-2}(K) + b_{m-1}(X; \mathbb{Z}_2) - 1.$$

We will now use the ideas of Grünbaum [1] to show that this theorem can be significantly improved when the hypotheses are strengthened somewhat.

THEOREM B. *If $X$ is any closed triangulable manifold ($m \geq 3$), then $\mathrm{ch}_{m-2}(X) \leq 6$. Further if $K$ is any subcomplex of a triangulation of $X$ and contains at least one $(m-2)$-simplex, then $m\alpha_{m-1}(K) < 6\alpha_{m-2}(K)$.*

*Proof.* The first part will follow from the second (as in the proof of Theorem 2 of [2], for example). Let $K$ be a subcomplex of a triangulation $L$ of $X$ and let $\sigma_1, \sigma_2, \ldots, \sigma_t$ be the $(m-3)$-simplices of $K$ which are incident to at least one $(m-2)$-simplex of $K$. Since $X$ is an $m$-manifold ($m \geq 3$), $Lk_L\sigma_i$, $1 \leq i \leq t$, is a triangulation of the 2-sphere $S^2$. Further $Lk_K\sigma_i$, $1 \leq i \leq t$, is a subcomplex of $Lk_L\sigma_i$ and contains at least one vertex.

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Applying the case $m = 2$ of Theorem A (or Lemma 1 of [1]) one gets

$$\alpha_1(Lk_K\sigma_i) \leq 3\,\alpha_0(Lk_K\sigma_i) - 3, \quad 1 \leq i \leq t.$$

Adding these inequalities one has

$$\binom{m}{m-2}\alpha_{m-1}(K) \leq 3\binom{m-1}{m-2}\alpha_{m-2}(K) - 3t$$

which implies $m\alpha_{m-1}(K) < 6\alpha_{m-2}(K)$.

Thus the "finiteness theorem" stated in the introduction of [2] can be improved to the above "six color theorem"; however the above proof does not generalize to pseudomanifolds $X$.

For any compact triangulable space $X$ let us denote by $\mathrm{Ch}_i(X)$ the least number of colors which suffice to label the $i$-simplices of any triangulation of $X$ in such a way that distinct faces of an $(i + 1)$-simplex are assigned distinct labels. It is clear that $\mathrm{ch}_i(X) \leq \mathrm{Ch}_i(X)$. We can use Grünbaum's trick of using "weight functions" (see [1]) to supplement Theorem B with the further assertion that *for any closed manifold $X$ of dimension $m \geq 3$, $\mathrm{Ch}_{m-2}(X) \leq 6(m - 1)$*. The same trick and Theorem A can be used to get upper bounds for $\mathrm{Ch}_{m-2}(X)$ when $X$ is an $m$-dimensional pseudomanifold.

*Further results and conjectures.* We have proved that if $X$ is a compact triangulable space with dimension greater than or equal to $2i + 3$, then $\mathrm{ch}_i(X) = \infty$. Another result of some interest is that $\mathrm{ch}_{m-1}(X) = 2$ whenever $X$ is a closed manifold with dimension $m \geq 2$. We hope to give elsewhere a proof of the fact that $\mathrm{ch}_i(X)$ is finite whenever $X$ is a closed manifold with dimension less than or equal to $2i + 2$. In view of Theorem B above it seems likely that the number $\mathrm{ch}_{m-2}(X)$ is the same for all closed $m$-dimensional manifolds $X$ with $m \geq 3$; quite possibly the numbers $\mathrm{ch}_{n-1}(M^{2n})$ are the only ones which reflect the global topology of a closed manifold.

If $X$ is a closed triangulable $m$-manifold ($m \geq 3$), then $\mathrm{ch}_{m-2}(X) \leq 4$: this improvement of the first part of Theorem B can be obtained by using the four color theorem.

*Added in proof.* For more discussion regarding results mentioned above see [3] and [4].

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