II. *Comment by C. D. Olds, San Jose State College.* For readers who wonder how the computer value reported by Klein might be obtained, the following manipulations (easily justified) may be of interest.

$$I = \int_0^1 x^x dx = \int_0^1 e^{x \ln x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!} dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (x \ln x)^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1} n!} \int_0^{\infty} e^{-t} t^n dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1} n!} \Gamma(n+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} = 0.78343051 \cdots .$$

The series is particularly attractive because of its rapid convergence.

A Condition which Implies Monotonicity

E 2246 [1970, 653]. *Proposed by Andrew Rochman, Saint Louis University*

Let f be a nonconstant, real valued, continuous function such that, for all $x, y \in R$, $f(x+y) = \phi(f(x), y)$. Prove that f is monotonic.

I. *Solution by G. A. Heuer, Concordia College.* Indeed, f is strictly monotone or constant. For, suppose $f(a) = f(b)$ for some a, b with $b-a=h>0$. Then $f(a+t) = \phi(f(a), t) = \phi(f(b), t) = f(b+t)$ for all t; i.e., f is periodic with period h. Since $f(a) = f(b)$, there is a point c in (a, b) such that $f(c)$ is an extreme value on $[a, b]$. If $\epsilon > 0$ is chosen so that $c-\epsilon$ and $c+\epsilon$ are in $[a, b]$, then $f(x) - f(x+\epsilon)$ has opposite signs at $x=c$ and $x=c-\epsilon$, so is zero at some point a_0 between; thus $f(a_0) = f(a_0+\epsilon)$. But this implies f is periodic with period ϵ. Hence f is constant.

II. *Solution by K. S. Sarkaria, State University of New York at Stony Brook.* Suppose f is not monotonic; then we can find three points $x_1 < x_2 < x_3$ such that $f(x_1) > f(x_2) < f(x_3)$ or $f(x_1) < f(x_2) > f(x_3)$. We can assume the first case since the proof for the second case is similar.

Now choose a real number A so that $f(x_2) < A < \min\{f(x_1), f(x_3)\}$, and define a and b by

$$a = \text{lub}\{x \mid f(x) = A, x < x_2\}, \qquad b = \text{glb}\{x \mid f(x) = A, x > x_2\}.$$

Obviously we have $x_1 \leqq a < x_2 < b \leqq x_3$ and the function $f(x)$ is bounded above by A in $[a, b]$. Further, since $f(a) = f(b) = A$, it follows that $f(x+b-a) = \phi(f(b), x-a) = \phi(f(a), x-a) = f(x)$, for all x. Hence $f(x)$ is periodic with period $b-a$ and so is bounded above by A for all values of x. This is not possible.

Also solved by J. Aczel, B. H. Aupetit, Donald Batman, R. L. Enison, Michael Greening, Eric Langford, Harry Lass, William Margolis, Robert Patenaude, M. A. Radke, Wayne Roberts, Sid Spital & Peter Fowler, D. P. Stanford, Walter Stromquist, and B. L. D. Thorpe.

Professor Aczel notes that the problem is not new and that its solution can be found in Aczel, Kalmar and Mikusinski, *Sur l'équation de translation*, Studia Math. 12 (1951), 112–116. The problem is also solved in his book, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1966, p. 19. He has generalized the problem in a later paper, *On strict monotonicity of continuous solutions of certain types of functional equations*, Canad. Math. Bull., 9 (1966), 229–232.

Representation of Integers in Terms of a Given Set

E 2247 [1970, 765]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $a_1, \cdots, a_n$ be a set of relatively prime positive integers. Let $F(a_1, \cdots, a_n)$ represent the largest integer which cannot be represented in the form $c_1a_1+c_2a_2+\cdots+c_na_n$, where $c_1, \cdots, c_n$ are nonnegative integers. Prove the following:

(1) If $(a, b)=1$, $a>0$, $b>0$, and c is a positive integer such that c is nonrepresentable in the form $Aa+Bb$, with A, B nonnegative integers, then $F(a, b, c)<F(a, b)$.

(2) If $(a, b)=1$ and $2<a<b$ and t is an integer such that $ta<b<(t+1)a$, then $F(a, b, ab-(t+1)a-b)=ab-a-2b$.

Solution by the proposer. Let m and n be positive integers such that $(m, n)=1$.

LEMMA. *If A and B are integers such that $A+B=mn-m-n$ then exactly one of A and B is representable in terms of m and n.*

Proof: If both A and B were representable then $A=rm+sn$, $B=tm+un$, so that $mn-m-n=(r+t)m+(s+u)n$. Hence $mn=(r+t+1)m+(s+u+1)n$. This implies $(s+u+1)\equiv 0 \bmod m$ and $(r+t+1)\equiv 0 \bmod n$. But since $s+u+1>0$ and $r+t+1>0$, we have $s+u+1\geqq m$ and $r+t+1\geqq n$ so that $mn=(r+t+1)m+(s+u+1)n\geqq 2mn$, a contradiction. Now suppose A is not representable. Since $(m, n)=1$, $A=Rm-Sn$ where $S>0$ and $0\leqq R\leqq n-1$. Hence $B=mn-m-n-A=mn-m-n-Rm+Sn=(n-R-1)m+(S-1)n$ so that B is representable.

COROLLARY. *Since 0 is representable but no negative integer is representable,* $F(m, n)=mn-m-n$.

Proof of (1): Let $(a, b)=1$. If c is not representable in terms of a and b, then by the lemma, $ab-a-b-c$ is representable, i.e., $ab-a-b-c=ra+sb$ where r and s are nonnegative integers. Therefore $ab-a-b=ra+sb+c$. It follows that $F(a, b, c)<F(a, b)$.

Proof of (2): By repeated applications of the lemma it is seen that the following integers are not representable in terms of a and b: $ab-a-b$, $ab-2a-b$, $\cdots$, $ab-ta-b$, $ab-a-2b$. Now $ab-ra-b=\{ab-(t+1)a-b\}+(t+1-r)a+0b$ is representable in terms of a, b, $\{ab-(t+1)a-b\}$ for $0\leqq r\leqq t+1$. However, if $ab-a-2b$ were also so representable, then $ab-a-2b=Aa+Bb+C\{ab-(t+1)a-b\}$ with A and B nonnegative. Now $C\neq 0$ because $ab-a-2b$ is not representable in terms of a and b alone. On the other hand $C=0$ is implied by $ta<b$ whence $ab-a-2b<ab-(t+1)a-b$. From this contradiction the desired conclusion follows.

Also solved by Paul Fan, N. Felsinger, M. G. Greening (Australia), and Simeon Reich (Israel).