GROUPS OF ODD ORDER

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Abstract. The Feit-Thompson theorem, that a group of odd order is soluble, was always a challenge to those who believed that beautiful theorems should have beautiful proofs.

1. History and Strategy

Group theory and Galois theory developed together two centuries ago in the search for solubility. It was around 150 years later that Feit and Thompson proved their famous theorem [1]

Theorem 1. *(Feit-Thompson) All groups of odd order are soluble.*

This, in its time, was the starting point for the mammoth collective effort by group theorists to find all finite simple groups. For a historical account of this subject see [2].

In this note I will present a proof of theorem 1 inspired by ideas of Emil Artin. Ultimately, a group of odd order is soluble because a real polynomial of odd degree has a real root. This polynomial will appear as the characteristic polynomial of a matrix over a real field.

I will now outline the strategy of the proof of Theorem 1. This is to construct a large family of complex characters (homomorphisms to $\mathbb{C}^*$) of the group G (of odd order N) and then show that they cannot all be trivial. This, formulated in Section 4 as Theorem 2, easily leads to Theorem 1.

These characters are constructed in Section 5 from determinants of representative matrices which arise from Artin's idea of considering the action of a finite group G on all its subsets and then using algebra extensions of $\mathbb{Q}$. Complex characters of G are best understood through the *unitarian* trick of Hermann Weyl, which applies to compact Lie groups and in particular to finite groups.

We will use both algebraic numbers and algebraic functions. Note that an algebraic function defines an algebraic curve which in general consists of several irreducible curves. If there is only one, the functions form a field. Galois theory is traditionally defined only for fields. This is a much more delicate theory than the theory for algebras. In particular

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it depends importantly on the arithmetic of the integer N, the degree of the curve. The proof of Theorem 1, by Feit and Thompson, involves these arithmetic considerations and the way they are implemented in the structure of the group G. All of this is intimately related to Galois Theory. By contrast our approach, focusing on the algebras and ignoring irreducibilty questions is much cruder. It is precisely because it **ignores Galois Theory** that it leads to a simple proof of Theorem 1. But, in contrast with the Feit-Thompson proof, it yields no information about the internal structure of G. It says nothing about the way the Sylow subgroups for different primes are intertwined. Our proof demands less and delivers less. But it does give a short simple proof of Theorem 1, responding to the challenge in the Abstract, of **a beautiful proof of a beautiful Theorem**. As a philosophical digression let me give an analogy. If you, as the viewer, are asked to decide whether the animal in front of you is a camel or a dromedary, there are two ways of finding out. The **external** way is to count the number of humps, an easy task. The **internal** way is to examine the DNA of the two animals and find the genetic difference. This is obviously much harder, but it gives much more information. But counting humps is sufficient to answer the question. Feit and Thompson are the geneticists, while I just count humps.

Returning now to the strategy of the proof we note that this leads to our first tranche of characters and these will detect metabelian groups.

2. Algebraic Preliminaries

A **real number field** will mean an algebraic number field k embedded in the reals:

$$\mathbb{Q} \subset k \subset \mathbb{R}. \tag{2.1}$$

Similarly, a **complex number field** will mean an algebraic number field together with a choice of embedding in $\mathbb{C}$. If we start from a complex extension of $\mathbb{Q}$ this will lead to a complexification of (2.1). The case that will be basic to this paper is the extension $\mathbb{Q}(\rho) := \mathbb{Q}(\sqrt{-3})$ whose integers are called the Eisenstein integers E. This is a unique factorization domain and the fundamental units are (excluding 1 and making a choice of signs)

$$\rho = \frac{-1+\sqrt{-3}}{2} = \exp\frac{2\pi i}{3}, \quad \overline{\rho} = \frac{-1-\sqrt{-3}}{2} = \exp\frac{-2\pi i}{3}, \tag{2.2}$$

the solutions of the equation

$$u^2 + u + 1 = 0. \tag{2.3}$$

Finally, we discuss determinants. If V is a vector space of dimension N over the real number field k, the **determinant** is a homomorphism:

$$\det : \mathrm{Aut}(V) \to k^* \subset \mathbb{R}^* \tag{2.4}$$

Similarly, if V and the automorphism are defined over the complex number field $k(\rho)$, det takes values in $k(\rho)^* \subset \mathbb{C}^*$

We now pass to the *spin* double covering given by the **change of variable**

$$z = u^2 + u + 1. \tag{2.5}$$

This leads us to the real function field $K := k(u)$ and its complexification $K(\rho) := k(\rho)(u)$, where u is a variable. Such functions can be evaluated at complex points (or numbers) and yield complex numbers in the relevant field. For K, evaluation at ρ or $\sqrt{\rho} = -\rho^2$ gives values in $k(\rho)$.

On the Riemann surface defined by (2.5), functions can be even or odd under the involution

$$u \mapsto -u. \tag{2.6}$$

and there is (see [10])

(2.7) a distinguished spin structure

which can be even or odd according to the value of

(2.8) the Arf invariant of a quadratic function

As a module over $k(z)$, K has rank 2 and so endomorphisms of K can be viewed as 2×2 matrices over $k(z)$: a non-commutative algebra.

In terms of groups of invertible elements we have

$$\mathrm{End}(K)^* = k(u)^* \ltimes (\pm 1) \tag{2.9}$$

is a semi-direct product with the involution $u \mapsto \overline{u}$ on $k(u)^*$. Thus we see $\mathrm{End}(K)^*$ as a subgroup of index 2 in the unimodular elements of the matrix algebra; the two choices correspond to $\mathbb{C}^+$ and $\mathbb{C}^-$, the two halves of the complex plane minus the real line. Note: Geometers may recognize here the two classes of projective bundles with fibre $P_1(\mathbb{C})$ determined by the parity of the first Chern class i.e. by the second Stiefel-Whitney class, whose vanishing gives spin.

The distinction between odd and even functions, under the involution $u \mapsto -u$ extends to vectors, matrices and eigenvalues.

3. Determinants

Since K is a lift of $k(z)$ any real matrix $A \in GL(N, K)$ can be viewed as a matrix in $GL(2N, k(z))$, which has as eigenvalues the $2N$ complex conjugate variables:

$$(\lambda_1, \dots, \lambda_N) \text{ and } (\overline{\lambda_1}, \dots, \overline{\lambda_N})$$

The λ_l and $\overline{\lambda_l}$ are distinguished by choosing

$$\lambda_l \in \mathbb{C}^+ \text{ and } \overline{\lambda_l} \in \mathbb{C}^-.$$

Taking the determinant of A then gives a homomorphism

$$\det : GL(N, K) \to K^*.$$

This will now be spelled out in more detail in terms of eigenvalues. Note that $\zeta = \exp \frac{\pi i}{N}$, generates the cyclic group of order $2N$ which separates into odd and even powers. The odd powers are never equal to $+1$ and so, since N is odd, we have

$$\zeta^N = -1. \tag{3.1}$$

The fundamental units of K_N, the invertible $N \times N$ matrices over K, have eigenvalues in the upper half-plane

$$\sqrt{\rho}\zeta^r \text{ where } \zeta = \exp \frac{\pi i}{N} \text{ and } 1 \leq r \leq N. \tag{3.2}$$

Here ρ is the primitive element in the field $k(\rho)$ and ζ is a complex value of the variable u. Figure 1 below shows how these two numbers are related and how the variable z in (3.3) corresponds to the parameter along the chord joining them.

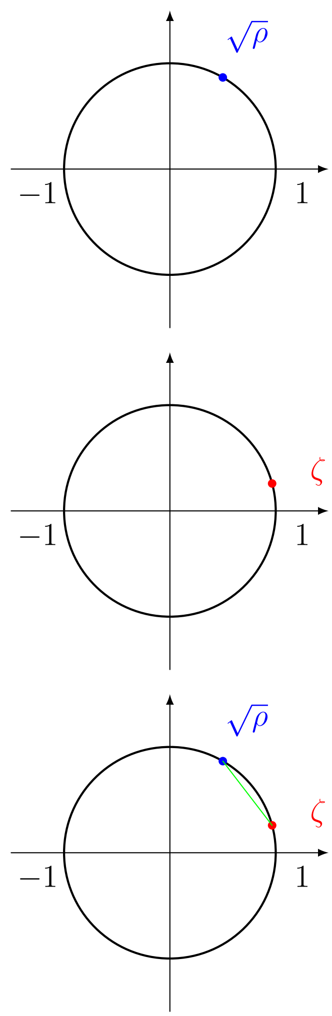

FIGURE 1. Relation between the numbers $\sqrt{\rho}$ and ζ in the complex plane $\mathbb{C}$.

Given an ordered basis for a vector space, automorphisms can be expressed as matrices and this is natural in our context since our group G will provide bases. Note that determinants are unchanged by any even permutation of the basis.

A complex $N \times N$ matrix A, with N odd, has N complex eigenvalues. If A is real, i.e. $A = \overline{A}$, then the characteristic polynomial

$$\varphi(A, z) := \det(A - zI) \tag{3.3}$$

is real and has odd degree N. Hence it must have a real root, giving a real eigenvalue λ_1 of A.

The problem of finding a real root λ_1 is unexpectedly deep. While algorithms do exist, they are not robust, so that small variations in the data (the coefficients) can lead to hopping around the roots. This is all relevant to Theorem 1 if one wants to find a soluble chain of field extensions. This is a difficult task precisely because of the *embarras du choix* but, for our purposes, it can be ignored.

As noted after (2.4), a complex matrix A over the complex field $k(\rho)$ has, when non-singular, a complex determinant $\det A$ which gives a homomorphism of groups:

$$\det : GL(N, k(\rho)) \to k(\rho)^*. \tag{3.4}$$

The usual formula $|z|^2 = z\bar{z}$ becomes in the matrix version

$$||A||^2 = \det A \det \overline{A} \tag{3.5}$$

The complex determinant $\det A$ is the product of all the eigenvalues:

$$\det A = \prod_{l=1}^{N} \lambda_l. \tag{3.6}$$

where we choose λ_l (as opposed to its conjugate $\overline{\lambda_l}$) by the same convention as before so that $\lambda_l \in \mathbb{C}^+$.

Letting A be unitary and replacing A by $A - zI$, with z a variable, we get a homomorphism

$$\det : GL(N, k(\rho)(z)) \to k(\rho)(z)^*. \tag{3.7}$$

We can specialize the variable z to any value which is **not** an eigenvalue of $A \in GL(N, k(\rho))$, since we need a non-zero determinant. In fact, after the lift to K, $z = u^2 + u + 1$ and we can take $u = \sqrt{\rho}$ because the eigenvalues of the matrix $(A - \sqrt{\rho}I)$ are, using (3.2):

$$\lambda - \sqrt{\rho}\zeta^r \text{ where } \lambda^N = 1. \tag{3.8}$$

and, because N is odd, none of these is zero. See Figure 2 and 3. The calculation of δ_3 comes from

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} \tag{3.9}$$

that measures the gap on the unit circle between i and $\sqrt{\rho}$, comparing Gaussian and Eisenstein integers. Note that the 3 points $-1, \sqrt{\rho}, i$, and the chord joining i and $\sqrt{\rho}$ are all in the closed unit disc $|\lambda| \leq 1$, and correspond to the 3 fractions occurring in (3.9). Replacing 3 by 5, so that regular triangles become regular pentagons is illustrated by Figure 3. Again we have even and odd pentagons. Equations such as (2.5) and its generalizations give double coverings branched at the points of an **even** polygon with an **odd** number of sides, like 3,5. The points of the odd polygons are clearly distinct from those of the even polygons. When we start iterating these constructions, as we will do in section 5, we will have a sequence of branch points w_j indexed by j. In passing from j to $j+1$ we will have two sets of branch points; the "old" ones labelled by j and the "new" ones labelled by $j+1$. We want to avoid any new ones coinciding with any old ones and this we achieve by our separation into odd and even. We have used another symbol u instead of w, since we want u to be taken successively as a w_j, enabling our argument to be an inductive one, with the fields and the branch points being labelled accordingly. Note that u_1 is the solution of equation (2.5). This explanation is designed to help the reader see through the formalism of section 5, which may otherwise look baffling.

Hence $A - \sqrt{\rho}I$ is non-singular and its determinant gives the odd character

$$\varphi : GL(N, k(\rho)) \to K^*(\sqrt{\rho}). \tag{3.10}$$

We will elaborate on this in the next section. Note that the complex numbers (3.8) are inside but not on the unit circle, so the character (3.10) is *not unimodular*. This comes from the fact that, in Figures 2 and 3 (with $N = 3$), $i - \sqrt{\rho}$ is the chord (straight line) from i to $\sqrt{\rho}$ and not the arc of the circle with angle $\pi/6$ occurring in (3.9). The essential point is that the circle is convex (see section 12 of [11] for an extended discussion of convexity in compact Lie groups).

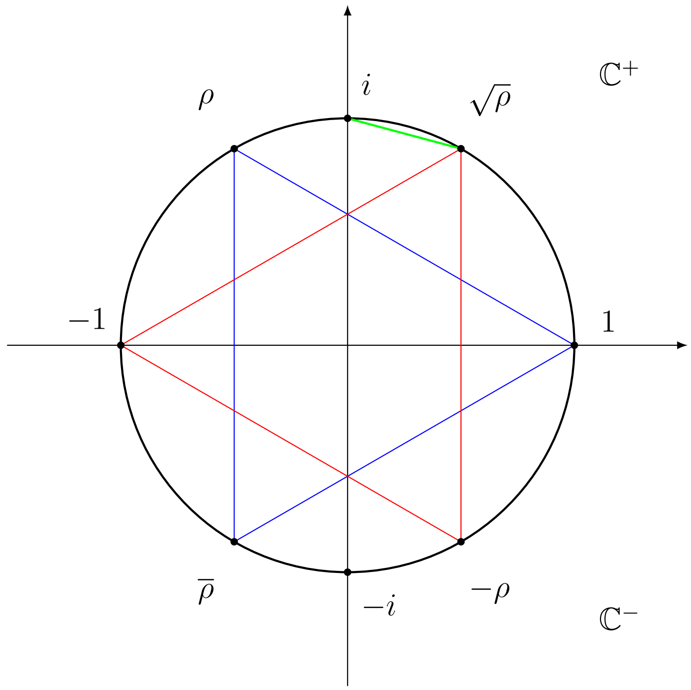

FIGURE 2. For $N = 3$, detail of the unit circle together with the relevant units and the chord whose mid-point is at a distance from the arc of $\delta_3 = 1 - \frac{\sqrt{3}}{2}$. The odd triangle (blue) corresponds to ρ and the even triangle (red) corresponds to $\sqrt{\rho}$.

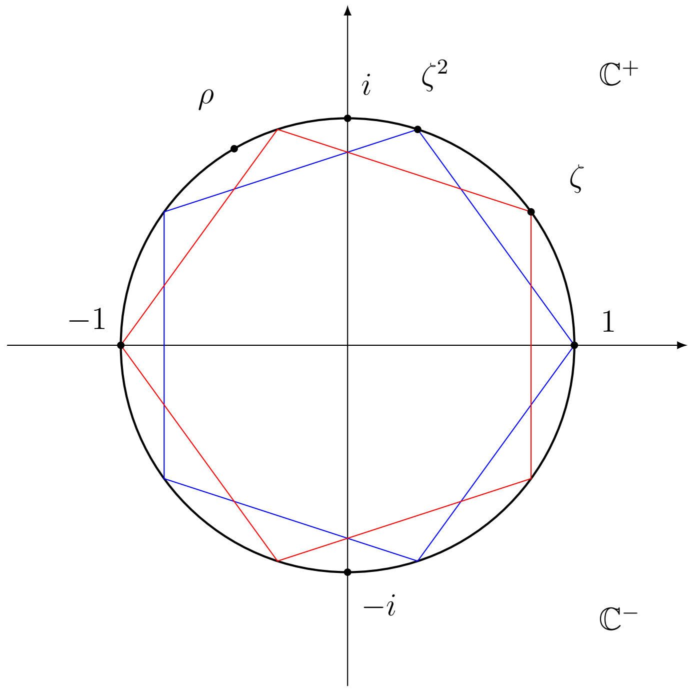

FIGURE 3. For $N = 5$, detail of the unit circle together with the relevant units; this diagram will give the distance δ_5. More generally $\delta_N = 1 - \sin \frac{\pi}{N}$. As in the previous figure, the red pentagon is even and the blue pentagon is odd, for $N = 5$.

We will use two different symbols, n and N with $n \leq N$ to distinguish between the order n of ρ and the order N of ζ^2.

When we consider a finite group G, as we will do in the section below, N will be the order of G and n the exponent of G, i.e. the least n so that $g^n = 1$ for all $g \in G$.

4. GROUPS AND SETS

Section 2 was a review of matrices and determinants and section 3 was a review of determinants. Now we will start from an abstract group G of odd order N and, following Artin, view it as acting on the set S of its elements. It acts via permutations σ and, since N is odd, $\text{sign}(\sigma) = 1$ and G acts on S via the alternating subgroup of the symmetric group.

A finite set S with a chosen element s_1 (the base-point) is a **based set**. If there is a (non-trivial) involution $s \mapsto s^{-1}$ fixing the base point, we call S a **symmetric set**. G acts on S by a left action $s \to \sigma s$ and a right action $s \to s\sigma^{-1}$. Both actions together $s \to \sigma s \sigma^{-1}$ give the **conjugation action** which preserves the symmetric structure of S, where the involution is group inversion and the identity 1 is the base point. The centre $Z(G)$ is the kernel of the action of G and the orbits are the conjugacy classes. Conjugacy classes are called **real** if fixed by inversion and **complex conjugate pairs** otherwise.

If S is any symmetric G-set (i.e. one whose G-action preserves the symmetry), we denote by S^* the set S with base-point s_1 removed. The set of all subsets of S is denoted by 2^S with cardinality $|2^S| = 2^{|S|}$. There is one distinguished subset, namely the empty set $\varnothing$. Removing this from 2^S leaves us with the set $(2^S)^*$ of **non-empty subsets** with cardinality $2^{|S|}-1$. Note that if $|S| \neq 0$ then $2^{|S|}-1 \neq 0$ and is **odd**. In fact if $|S| \geq 3$ then $2^{|S|}-1 \geq 7$.

In addition to $\varnothing$ there is another equally distinguished subset of S, namely the whole set Ω. There is a duality (taking complements) which interchanges $\varnothing$ with Ω.

The theory could be pursued entirely in the framework of finite sets and Boolean algebra and this was essentially Artin's idea. In fact, to connect with field theory, we shift now to linear algebra using matrices and determinants.

If S is a basis of a real or complex vector space V, then 2^S is a basis of the exterior algebra $\wedge^\bullet(V)$ with the empty set $\varnothing$ a basis of $\wedge^0(V)$ and the whole set Ω the basis of $\wedge^N(V)$ where $N = |S|$. If V is a k-vector space then, as explained in Section 2, we get the odd character φ of (3.10).

Keeping track of the parity of a character of G becomes quite delicate as we carry on this iteration. The delicacy comes from the Arf invariant of (2.8). But there is an alternative and easier way of showing non-triviality of a complex character and that is to show that it is *not unimodular*. This is what we shall actually do based on the remarks at the end of section 3, after formula (3.9).

5. The Iterative Process

Having started the Artin process of using the conjugation action of G on its non-empty subsets we now plan to iterate this process N times. As explained at the end of Section 3, the purpose of the iteration is to handle groups of any odd exponent $n \leq N$. The index j increases with the exponent n but eventually stops at N.

Our iterative process will produce a finite sequence, indexed by j (with $1 \leq j \leq N$), of

$$\text{odd integers } N_j, \text{ with } N_1 = N = |G|, \quad N_{j+1} = 2^{N_j} - 1 \tag{5.1}$$

$$\text{sets } S_j \text{ with } |S_j| = N_j, S_1 = G, \quad S_{j+1} = (2^{S_j})^* \tag{5.2}$$

$$\text{real fields } k_j \text{ with } k_1 = \mathbb{Q} \text{ and } k_{j+1} = N_j \times N_j \text{ matrices over } k_j \tag{5.3}$$

so that k_{j+1}^* consists of the invertible scalar $N_j \times N_j$ matrices over k_j. We also need to iterate the extraction of square roots defining the sequence of variables w_j by

$$w_j = w_{j+1}^2 + w_{j+1} + 1. \tag{5.4}$$

$$\text{function fields } K_j \tag{5.5}$$

$$\text{linked by translation } K_j(w_j) = K_{j-1}(w_{j-1} - \sqrt{\rho_{j-1}}) \text{ for } j > 1 \tag{5.6}$$

where ρ_{j+1} is a root of the polynomial on the right hand side of (5.6),

$$\text{modules } V_j = V(S_j) \text{ over the fields in (5.3) to (5.6)} \tag{5.7}$$

$$\text{volume elements } \Omega_j \in \wedge^{N_j}(V_j). \tag{5.8}$$

The invertible elements of each algebra act on the volume element by the determinant

$$\det : GL(N_j, K_j) \to K_j^*, \tag{5.9}$$

giving a character by the formula

$$\psi_j(A_j) = \det(A_j - \sqrt{\rho_j} I). \tag{5.10}$$

where the ρ_j are defined in (5.6).

The translation $w_j \to w_j - \sqrt{\rho_j}$ induces an affine map

$$\alpha_j : K_j \to K_{j-1} \tag{5.11}$$

that is consistent with the characters ψ_j, so that we have the commutative diagram:

$$\begin{array}{ccc} GL(N_j, K_j) & \longrightarrow & GL(N_{j-1}, K_{j-1}) \\ \downarrow \psi_j & & \downarrow \psi_{j-1} \\ K_j^* & \xrightarrow{\alpha_j} & K_{j-1}^* \end{array} \tag{5.12}$$

where the top map is a consequence of the Cayley-Hamilton theorem applied in our context.

At the end of section 3 we explained the rationale of the complicated looking inductive formulae above. There is one further point that the reader might find helpful and that concerns the commutative diagram (5.12) and the reference to the Cayley-Hamilton Theorem. The inductive step from j to $j+1$ involves replacing a vector space by its exterior algebra, and an $m \times m$ matrix A by the $\binom{m}{r} \times \binom{m}{r}$ matrices $A(r)$ acting on the r-th exterior power. For any scalar z, the translated matrix $A - zI$ (with I the unit matrix) then acts on the exterior algebra as

$$\sum_r (-z)^{(m-r)} A(r) \tag{5.13}$$

The determinant of this action is just the characteristic polynomial

$$\det(A - zI) \tag{5.14}$$

The Cayley-Hamilton Theorem asserts that, replacing the scalar z by the matrix A in (5.13) gives zero. Of course the sum (5.13) or the determinant in (5.14) are now traces or determinants of much larger matrices. In (5.14) the matrices go from $m \times m$ to $2^m \times 2^m$. In modern notation the Cayley-Hamilton Theorem looks obvious because $A - AI = 0$. But modern algebra was essentially created by Hamilton and Cayley to make a deep fact seem obvious. A sophisticated way of interpreting the Cayley-Hamilton Theorem is to say that a projective module and its Koszul resolution, by the exterior algebra, define equivalent elements of K-theory over the ground ring.

This explains the commutative diagram (5.12), remembering that $r = 0$ indexes the module being resolved. Note that the index j in these formulae is naturally decreasing, involving a downward (finite) descent starting from N. The indexing of the variable u ends with u_0 corresponding to the empty set (this is the sheaf of the point that is being resolved). The shift to w_1 corresponds to the free module defined by V.

Thus we have indeed constructed a sequence of N complex characters and our aim is to show that they cannot all be trivial. In fact, we will show that the character ψ_N of G is not unimodular and hence is non-trivial.

The action of G on the vector space V_N over the complex function field K_N might not be unitary for any metric though it might still be unimodular, i.e. with determinant 1. We will show that this does not happen. The reason is that at least one of the eigenvalues λ of the relevant matrix has modulus *strictly less than* 1. This is clear from the geometry of Figure 2 and 3, showing that the chord is inside the circle. The deviation from 1 is extremely small, a crude asymptotic estimate of the lower bound is

$$1 - |\lambda| \sim 2^{-N_j} \text{ for large } j. \tag{5.15}$$

The determinant is therefore just less than 1, and the character ψ_N is non-trivial. This is what we set out to prove.

Thus we have now proved

Theorem 2. *A group G of odd order has a non-trivial complex character.*

Theorem 1 is an easy consequence of Theorem 2 as we will now show. Suppose Theorem 1 is false, then there must be a group G of minimal odd order N which is not soluble. Applying Theorem 2 to G leads to an exact sequence of groups

$$1 \to H_1 \to G \to H_2 \to 1 \tag{5.16}$$

where $|H_1|$ and $|H_2|$ are both less than N and so soluble (by the minimality assumption: in fact $H_2 \subset \mathbb{C}^*$ is necessarily abelian). But then (5.16) gives a chain of subgroups of G, each normal in its successor, and with abelian quotient. This is a definition of solubility and gives the required contradiction, thus establishing Theorem 1.

This completes the formal proof of Theorem 1. In the final section to follow I will make comments on the nature of the proof and discuss its implications.

6. Comments

In this paper I have presented a short proof of the Feit-Thompson theorem, using only elementary ideas of linear algebra and number theory. The main novelty was the use of an iterative process based on ideas of Artin and Hermann Weyl. For simplicity I avoided more sophisticated ideas and stayed within the *lingua franca* common to all mathematicians and physicists (except for explanatory remarks that strayed into modern commutative algebra). I did this because it seems likely that the ideas of this paper will apply to a much larger class of problems, drawn from geometry, number theory and physics.

To prove the Feit Thompson Theorem with no arithmetic information about the order N, we had to use the very large numbers $M(N)$ and the extremely small numerical bounds (5.15). This programme can obviously be refined in various ways as briefly indicated below:

5.1 The number of times we have to exponentiate, replacing N by 2^N, is the exponent n of the group G, which may well be much smaller than its order. This leads to the whole arena of Burnside type problems related to the work of Zuk.

5.2 Integers with many prime factors are rare, so probabilistic methods can yield much better bounds.

The fact that $|G|$ is odd was used in several different ways, but the general strategy should still work for groups of even order, shedding light on the structure of all finite groups. In particular it should give a better understanding of the whole programme described in [2].

In number theory, Fermat's Last Theorem, famously proved by Andrew Wiles, is another challenge for those seeking simple proofs. The ideas of this paper offer hope for this task, as implied by my Fudan University lecture.

In physics the iteration process we used involves rescaling and leads to fractals and renormalization. The very small numbers in (5.15) are important for theory but not in experiment, where they can be ignored.

The famous Dirac *sea* of particles and anti-particles has energy levels modelled here by positive and negative powers of 2, when N is expressed dyadically. Also related to quantum mechanics is the fact that the points on the unit circle corresponding to the roots of unity, used in our field extensions, define convex polygons in a similar fashion to [9].

I hope to illustrate this in further publications with younger colleagues. But several earlier papers of mine have already utilized the ideas in various contexts. The non-existence of a complex structure on the 6-sphere is treated in two separate papers [4] [6]. An application in chemistry involving the remarkable elements Helium 4 and Helium 3 is described in [3].

There are important problems in algebraic topology that are relevant. The first is the now ancient solution of the Hopf invariant problem by J.F. Adams and the subsequent short proof in my paper with Adams [7]. More recently, a similar but deeper theorem about the Kervaire invariant has been (almost) completely solved by Hill, Hopkins and Ravenel [8]. It is probable that the methods of the present paper will similarly lead to a shorter proof.

Paper [8] arose from ideas of string theories and I anticipate significant applications in this direction. My short tentative paper with Greg Moore [5] will, I hope, fit naturally into this framework.

The Artin process leads to clusters and its iteration leads to micro-clusters, much studied in the physical sciences, indicating that our models provide a good framework at all scales.

Finally I shall comment on finiteness. The proof of Theorem 1 used, in an essential way, the finiteness of the order N of the group. It will be very interesting to investigate what happens when we allow N to grow to infinity. In number theory studying what happens as $N \to \infty$ has been the fundamental problem since the time of Euler and Riemann but, as is well known, much sharper estimates are now needed.

In physics, keeping N finite involves an energy cut-off and letting $N \to \infty$ leads to difficult unsolved conceptual problems.

It seems clear that serious logic is involved in this limit process (both in number theory and in physics). This brings us back to the great controversies of 100 years ago between Brouwer, Hilbert, Weyl and Gödel. Ultimately it depends on understanding the real numbers.

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