

# Geometric Models of Helium

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## Abstract

A previous paper [13] modelled atoms and their isotopes by complex algebraic surfaces, with the projective plane modelling Hydrogen. In this paper, models of the stable isotopes Helium-4 and Helium-3 are constructed.

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# 1 Introduction

For some time, I and my collaborators, notably Nick Manton, have been developing the idea that all types of matter can be modelled by 4-manifold geometry [1]. Simple particles, like the electron and proton, are described by some of the simplest oriented 4-manifolds. Because these particles are electrically charged, the manifolds are non-compact. Composites, starting with the Hydrogen atom, are modelled by more complicated manifolds. If the composite is electrically neutral, then the manifold is compact. Quantum numbers like baryon number and lepton number are related to the topological invariants of the manifold.

These ideas are an extension of earlier ones. For example, the Kaluza-Klein monopole (at a fixed time) is a smooth 4-dimensional Riemannian manifold that is interpreted as a particle in three dimensions [2, 3]. It is essentially the same as our model for one electron. Another inspiration comes from Skyrmons [4]. Skyrmons are solitons in 3-dimensional space, modelling protons and neutrons [5], and larger nuclei [6], and it was shown in [7] that Skyrmons can, to a good approximation, be extracted from the holonomy of Yang-Mills instantons in four dimensions.

The 4-manifolds provide static, classical models for matter. Dynamics arises from the dynamically-varying moduli of the manifolds [8, 9]. Also, attempts to model quantum mechanical features have been made [10, 11, 12]. Sections of suitable bundles over the 4-manifolds can model quantum states, capturing their energies and spins.

[13] proposed a dictionary which associated electrically neutral atoms to compact, complex algebraic surfaces [14, 15, 16]. The proton number  $P$  of the atom is associated with the topological invariant  $P' = \frac{1}{4}(e + \tau)$  of the algebraic surface, where  $e$  and  $\tau$  are the Euler number and the signature. This invariant is always an integer for algebraic surfaces. For  $P = 1$  we have Hydrogen and this is associated to the complex projective plane where  $e = 3$  and  $\tau = 1$ , so that  $P' = 1$ . From this meagre start it was remarkable how algebraic surfaces with higher values of  $P'$  seemed to be sufficiently plentiful to associate to atoms with higher values of  $P$ .

In the dictionary, isotopes with neutron number  $N$  correspond to algebraic surfaces with invariant  $N' = \frac{1}{4}(e - 3\tau)$ . Thus Deuterium has  $N = 1$  and Tritium has  $N = 2$ . The associated algebraic surfaces for ground states of these isotopes are obtained from the projective plane by blowing up one point for Deuterium and two points for Tritium. The location of the points blown up requires some symmetry breaking. The surfaces modelling the ground states are related, but have different intersection forms. The geometry also seems to indicate, to some degree, the stability of certain isotopes, not just for Hydrogen but also for higher proton numbers. [13] highlighted the case of Iron, where Iron-60 has

remarkable stability, in terms of its half-life. This is suggested by the geometrical model and not adequately explained by conventional methods of nuclear physics.

After Hydrogen, which accounts for 90% of the matter in the universe, the next most abundant element is Helium, which accounts for a further 9%, leaving only 1% for all the heavier elements. The most common isotope is Helium-4, with Helium-3 far less frequent.<sup>†</sup> These isotopes are stable and have many remarkable properties, depending on temperature and pressure.

Atomic Helium is particularly stable because its electrons form a closed shell, and the ionization energy (the energy to remove one electron) is higher than for any other atom. Also, the nucleus of Helium-4 is the alpha particle, a closed-shell (magic) nucleus. This means among other things that the energy to remove one proton or one neutron from the nucleus is very large, more than twice the typical energy required in other stable nuclei. On the one hand these properties make Helium-4 a most useful element, for example in cryogenics, but on the other, it poses severe tests for theoretical models.

The purpose of this paper is to describe in detail the algebraic surface  $M$  that should be associated to Helium-4. As we will show, it is beautiful, subtle and unique, fully matching the physical atom.

However,  $M$  escaped attention in [13]. Crucially,  $M$  is not simply-connected and its fundamental group is the quaternion group  $\Gamma$ , of order 8. The first Betti number is therefore  $b_1 = 0$ , but the first integer homology group is the abelianization of  $\Gamma$  and so is the product of two cyclic groups of order 2. The universal covering  $M^*$  of  $M$  has an action of  $\Gamma$  and the decomposition of the cohomology  $H(M^*, \mathbb{C})$  under this action gives by definition the cohomology of  $M$  with local coefficients. The group  $\Gamma$  has four irreducible 1-dimensional representations and one 2-dimensional one (see [17], Appendix).

$M$  is also fibered over an elliptic curve with a projective line as fibre, and no exceptional fibres. For topological reasons, one might expect such a fibration to have a holomorphic section. It does not, but the canonical line bundle  $K$  has order 4, so it does have a 4-fold section, giving four distinct points on each fibre, whose cross-ratio is harmonic. The monodromy action of the fundamental group  $\Gamma$  permutes the six values of the cross-ratio.

Symmetry here is so strong that it determines the model up to one overall scale factor  $k$  whose square models an energy level. The factor  $k$  itself depends on a choice of line bundle.

Our model is solvable in a strong sense, in that the partition function can be explicitly

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<sup>†</sup> The notation indicates that the proton number is 2 (Helium) and that the atomic mass number (baryon number) is, respectively, 4 or 3, so the neutron number is 2 or 1.

written down. In fact it can be identified with a classical Theta function. The basic reason for this solvability is that an elliptic curve or torus is flat. We can also base ourselves on the classical theory of Abel/Galois on the solution by radicals of quartic equations.

Having listed all the main properties of our algebraic surface  $M$  we will, in section 2, explain how to construct it and derive its properties. Sections 3 to 6 will explore it further. It is clear that  $M$  is indeed a very special algebraic surface, as befits a model for the very special atom Helium-4. Our conclusion is summarized in Section 7.

One can ask what atom is modelled by the value of the modulus associated with hexagonal symmetry. It is hardly a surprise that the atom in question is Helium-3, but this story is even more subtle, and deserves a whole section to itself, section 4.

The ideas in [13] need much further development. Proton numbers  $P = 1$  (Hydrogen) and  $P = 2$  (Helium) are, like rational and elliptic curves, rather special, while  $P > 2$  is the general case. This paper is therefore the transitional case, deserving particular treatment. The analogy with the theory of algebraic curves is in fact fundamental as will become clear later in the paper.

## 2 The Algebraic Surface $M(E)$

We will begin with a brief review of the classical theory of elliptic curves  $E$  over the complex numbers. The salient points are (up to isomorphism):

- (i)  $E$  is a 1-dimensional complex torus.
- (ii) Its fundamental group is  $\mathbb{Z} \times \mathbb{Z}$ .
- (iii) Its universal covering is  $\mathbb{C}$ .
- (iv) Its modulus is, up to the action of  $\text{SL}(2, \mathbb{Z})$ , a point in the complex upper-half plane.
- (v) The torsion subgroup, the symmetric group  $\Sigma_3$ , has a fixed point of order 2 (the square lattice) and two fixed points of order 3 (the hexagonal lattice).
- (vi) There are four spin structures on  $E$  corresponding to the four points of period 2 on the Jacobian  $J(E)$ ; 0 and the other three.
- (vii) At the spin level, the group  $\text{SL}(2, \mathbb{Z})$  is replaced by its double covering and the torsion subgroup becomes the even part of the binary octahedral group, and has order 24.
- (viii) Points (iv)-(vii) relate to the solvability of quartic equations developed by Abel and Galois 200 years ago.

Let us denote by  $E_n$  the moduli space of divisor classes of degree  $n$  on the elliptic curve  $E$ . All  $E_n$  are isomorphic but not canonically.  $E_1$  is the elliptic curve  $E$  itself. Choosing a base point, called 0, turns  $E$  into an abelian group  $E_0$ , the Jacobian  $J(E)$ . The space of unordered point-pairs on  $E$  is the symmetric square of  $E$  and we denote it by  $\text{Sym}^2(E)$ . By taking the divisor class of a point-pair we get a natural map  $f$  from  $\text{Sym}^2(E)$  to  $E_2$ , which is a principal homogeneous space of the Jacobian  $J(E)$ . Once a base-point pair has been chosen,  $E_2$  gets identified with  $J(E)$ . Moreover,  $f$  is a fibration

$$f : \text{Sym}^2(E) \rightarrow E_2 \tag{2.1}$$

with fibre a complex projective line.  $\text{Sym}^2(E)$ , the symmetric square of  $E$ , is the product  $E \times E$  divided by the involution that switches the two factors. The diagonal  $\Delta$  in the product, representing coincident pairs, is the branch locus of the involution. But it is not singular, because in the normal direction, a branch point for one complex variable is not a singularity (though it may be singular in a metric context).

$E$  can be realized as a double cover of a rational curve  $X$  branched at four points. This leads to the standard equation

$$y^2 - g(x) = 0, \tag{2.2}$$

where  $g$  is a quartic. This in turn leads to  $E \times E$  being a 4-fold cover of  $X \times X$  with four branch curves and  $\text{Sym}^2(E)$  as its symmetrization. We are interested in the most symmetric, quartic, with four roots forming a square on  $X$ . The simplest realisation is  $g(x) = 1 - x^4$ , with its roots at  $1, -1, i, -i$  having harmonic cross ratio  $\mu = -1$ . This corresponds to  $E$  being a square torus, for which the ratio of the periods (in the upper-half plane) is  $i$ .

$\text{Sym}^2(E)$  is close to being our model for Helium-4. However, the topology of symmetric powers of Riemann surfaces is well known [18], and the example  $\text{Sym}^2(E)$  has Euler number  $e = c_2 = 0$ . This is verified by considering the fibration  $f$  and noting that the base space, the torus  $E_2$ , has Euler number zero.

Our model algebraic surface  $M$  for Helium-4 involves a different fibration over  $E$ , which will be explained shortly. We still choose the most symmetric version of  $E$ , with cross ratio  $\mu = -1$ . Because of the close relation to  $E$ , from now on we will use the notation  $M(E)$  for  $M$ , our model algebraic surface for Helium-4. We must show that  $M(E)$  has the required invariants. What we need is to have  $e = 8$  and  $\tau = 0$ , so that  $P' = 2$  and  $N' = 2$ , agreeing with the proton/neutron numbers of Helium-4.

The dictionary set up in [13] did not assume that our models were simply-connected, but most of the examples we considered previously, like the projective plane, certainly were. One example had a non-trivial abelian fundamental group. Our putative model

$M(E)$  for Helium-4 has a finite, non-abelian fundamental group  $\Gamma$  and a compact universal covering  $M^*(E)$ . The dictionary should now be extended to such models by simply taking  $e = e(M^*(E))$ . This does not change the definition in [13] for simply-connected models, but as we shall see later it does give the desired formula  $e = 8$ . Since  $c_1 = -c_1(K)$  is a torsion class,  $(c_1)^2 = 0$ ,  $e$  is the only numerical invariant, and  $\tau = 0$ .

The cohomology of  $E \times E$  can be analysed by using the symmetric group as follows. Since  $E$  is a torus, its Poincaré polynomial is  $(1+t)^2$  and that of  $E \times E$  is  $(1+t)^4$ . This notation just uses the grading of cohomology. But the symmetric group  $\Sigma_4$  acts by permuting the four factors, so the Poincaré polynomial can be refined to take account of the representation theory of  $\Sigma_4$ . Over the complex numbers the representation on the complex cohomology is the  $\mathbb{Z}/2$  graded regular representation. The alternating group  $A(4)$  of even permutations preserves the  $\mathbb{Z}/2$  grading. The switch of the two  $E$  factors interchanges  $p$  and  $q$  in the Hodge  $(p, q)$  decomposition.

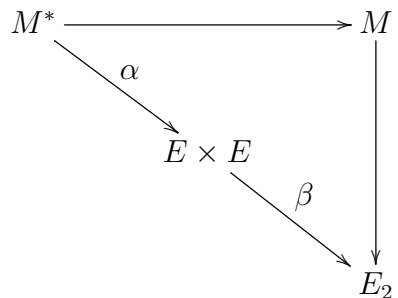
Now comes the further crucial refinement to the story which is the introduction of spin. See [24] for spin structures on curves of genus  $g \geq 1$ .  $A(4)$  is the even part of the octahedral group, and has order 12. It has a spin covering  $A(4)^*$  of order 24, the even part of the binary octahedral group. Its 2-Sylow subgroup is the quaternion group  $\Gamma$  of order 8.

This means that we can lift  $E \times E$  to a double cover, which we may denote by  $M^*(E)$ , and is the universal cover of  $M(E)$ .

We now need to choose an **odd** spin structure on  $E$ .  $\Gamma$  is the group of automorphisms of this spin manifold. We break the full  $A(4)^*$  symmetry by choosing one of the three non-trivial spin structures and the cyclic subgroup of order 3 then permutes the three choices.

From this classical algebra all the topological invariants emerge automatically. But for this we need to lift to the spinor world, and use Poincaré polynomials in  $u$  with  $u^2 = t$ .

We end this section by spelling out the Hodge diamond of our Helium-4 model  $M = M(E)$ . The topology of  $M = M(E)$  can be understood geometrically from the following diagram



where  $\Gamma$  (horizontal arrow) acts freely on  $M^*$  and  $M \rightarrow E_2$  is the  $P_1$  fibration. The diagonal maps  $\alpha, \beta$  are branched covers of degree 2 and 4 respectively. As we have seen, the action of the fundamental group  $\Gamma$  on  $H^*(M^*, \mathbb{C})$  is just its regular representation, and the action on the even-dimensional part is the regular representation of  $A(4)^*$ . Thus the Hodge diamond has non-zero entries only at the four vertices and in the middle: 1 at top and bottom, 1 on right and left and 4 in the middle. This gives the required value 2 for protons and 2 for neutrons.

The fact that  $\tau(M) = 0$ , verified above, is clear because  $\tau(M^*) = 0$ , and  $\tau$  is multiplicative for finite covers. This should be compared with the example studied in [25] of a fibration by curves of higher genus where the moduli vary and  $\tau \neq 0$ . This will be relevant when we come to study algebraic surfaces modelling atoms of higher baryon number.

Hovering over the diamond, like a kestrel, is the fundamental group of  $M(E)$ , the quaternion group  $\Gamma$  of order 8. This encodes its subtle spin structure.

We now return to explain in what way our fibration differs from that in (2.1). It differs by the fact that the fibres are **conics** and not lines, or equivalently that the projective fibration can be lifted to a rank 2 vector bundle with **odd**, rather than even  $c_1$ . The natural line bundle over a conic is the square of the natural line bundle over a line (the spin lift). Fixing the cross-ratio of 4 points on a conic rigidifies the conic and explains why the universal cover  $M^*(E)$  of  $M(E)$ , with group  $\Gamma$ , becomes isomorphic to  $E \times E'$  (where  $E'$  is a conjugate of  $E$ ). If we work arithmetically, as in the next section, then  $M(E)$  becomes isomorphic to  $M(E')$  after a Galois extension from  $Q$  to  $\Gamma$  (see section 6 on lattices).

We have given much more detail here than we really need, just to derive the numerical invariants of  $M$ , but this extra insight is both natural and useful. It shows that the calculations were essentially known to Abel and Galois. We are here treading the same path as Riemann who replaced the rigid geometry of Felix Klein by conformal geometry, while later he made the geometry more flexible by allowing a variable metric.

To sum up, our model manifold  $M$  for Helium-4 derives all its properties from classical algebra. Physicists know all this from basic abelian duality of circles and tori. The key advantage of our geometrical view is that modifications of the geometry in various ways can be easily understood, even when the original symmetry has been totally or partially broken.

There is a continuous abelian global symmetry but there is also a finite symmetry, related to spin, arising from the group  $\Gamma$ , and the ground state of Helium-4 is modelled by our symmetric quartic. Only the physical scale remains to be identified and this is where the model confronts experiment. Once this has been fixed, the dependence of the

energy of the excited states on the level  $k$  should provide a first approximation for the full discrete spectrum, with the scattering states to follow as with Hydrogen (see [12]).

### 3 The Arithmetic of the Helium-4 Model

Our Helium-4 model  $M(E)$  is naturally associated to an elliptic curve  $E$ , so the theory over the complex field can be refined to become a theory over the field of rational numbers  $\mathbb{Q}$ .

Naively one might think that physics only deals with real numbers and that arithmetic is irrelevant. This is true for classical physics but not for quantum physics. The first step down this path occurs when one replaces the real field  $\mathbb{R}$  by the complex field  $\mathbb{C}$  with all the subtleties associated to complex amplitudes, Bell's inequality and the probabilistic interpretation of the Copenhagen school. Arithmetic also enters when one counts discrete quantum states.

It was already noted in [13] that integers appeared through integer homology, and that the integral properties of quadratic forms over  $\mathbb{Q}$  played an important part in our models. When the models are algebraic surfaces, the period matrices give points in locally symmetric spaces, modulo arithmetic subgroups. This is classical for algebraic curves starting with elliptic curves. So it is natural to look for geometric models defined over  $\mathbb{Q}$  or some algebraic extension. This is the bread and butter of algebraic number theory, and elliptic curves play a central role at all levels.

Our model  $M$  of Helium-4 depends on an elliptic curve  $E$  and we should clearly start with the classical theory over  $\mathbb{Q}$ . There are different canonical forms for elliptic curves  $E$ , but they all depend on the fact that  $E$  is a double cover of a rational curve  $X$  branched over four points, which leads to a quartic equation. The most symmetric case is the curve we have been considering,

$$y^2 + x^4 - 1 = 0. \tag{3.1}$$

This curve gives the arithmetic model for  $M$ .

### 4 Helium-3

As promised we now return to Helium-3. Our aim is to show how our model explains the structure of Helium-3. Let us start with the known properties of physical Helium-3 atoms:



- (i) Helium-3 has one neutron less than Helium-4.
- (ii) Helium-3 is a fermion while Helium-4 is a boson.
- (iii) The ground state of the nucleus of Helium-4 has considerably lower energy than the ground state of Helium-3, but the electron energies are similar for the two isotopes.

For our models, property (iii) can be explained by a path in the moduli space of Helium-4 which raises energy. A small step along this path would model one neutron moving slightly out of the nucleus, decreasing the binding energy but still close by. A long way along the path the neutron would be on the verge of escaping and getting lost, leaving behind the fermionic Helium-3 nucleus. This happens eventually (mathematically at infinity). Two critical points are joined by a path along which energy is reduced.

This is a physical description but there is an explicit process that models it. Starting with our four branch points on  $X$  forming a square, pull one branch point far away so that the square looks like a kite. In the limit, one root of the quartic has become infinite, i.e. the leading coefficient has become zero, and our quartic has become a cubic. If we pull the branch point out of the plane symmetrically we end up with an equilateral triangle, the model for the ground state of Helium-3.

## 5 Elementary Geometry of $M(E)$

We will work in the Euclidean plane (but later we may prefer the hyperbolic plane in 3-space). Consider a rectangle whose sides have length  $a$  and  $b$ . By translation we can centre it at a fixed origin  $O$  and circumscribe an ellipse, centre  $O$ , with  $\frac{1}{\sqrt{2}}a$  and  $\frac{1}{\sqrt{2}}b$  as the lengths of the semi-axes. There are four choices (change orientation of both axes) unless the ellipse is a circle. The one parameter is the eccentricity  $\tilde{e}$  defined by

$$\tilde{e}^2 = 1 - \frac{b^2}{a^2} \quad (\text{if } a > b). \quad (5.1)$$

Let  $M(E)$  be the moduli space of such centred ellipses, but we broke the symmetry between the two factors, so  $M(E)$  inherits an ordering of pairs  $\{(+a, -a), (+b, -b)\}$ . This is part of its structure and corresponds to passing from the full group of even symmetries  $A(4)^*$  of a quartic to the 2-Sylow subgroup  $\Gamma$ .

With this description one can recognize various other models that have been studied, such as  $SU(2)$  BPS monopoles [19] and Skyrmons of charge 2 [20]. It would be instructive to examine these models further. The symmetries and the topology limit the number of models, so it is not surprising that the same moduli spaces appear, even when the energy

functions seem different. Conformally they have to coincide and the metrics only make a difference near the critical points. But crucially, spin is involved.

Flat space can be replaced by spaces of constant curvature, so monopoles in hyperbolic space fit the story, with the scalar curvature being now an additional parameter to add to all the others. Then  $M(E)$  would model Helium-4 in a weak gravitational field, arguably the real world, though the effect would normally be very small. This is the kind of modification discussed in [21] and would connect gravity with mass. Put succinctly, the mass of an atom of Helium would be raised slightly [23] by the weak gravity of the entire universe: a version of the philosophy of Mach.

## 6 Lattices

Our definition of the algebraic surfaces  $M(E)$  and  $M^*(E)$  was algebraic and has the advantage that it works over any field (except fields of characteristic 2). But over the complex numbers, elliptic curves, as explained at the beginning of section 2, have a transcendental description using the exponential function. The elliptic curve  $E$  is now derived from a lattice  $L$  in  $\mathbb{C}$  and the classical notation (see [22]) is to use  $z = x + iy$  as the complex variable and  $q = \exp(2\pi iz)$  as the modular variable. For the square torus  $z = i$ , so  $q = \exp(-2\pi)$ . Note that spinors require us to use the square-root,  $q^{1/2}$ .

Since all elliptic curves  $E$  have a flat constant curvature metric, our manifolds  $M(E)$  inherit flat metrics which minimize energy. But the **torsion** of the natural affine connection on  $M(E)$  models the topological "zero-point" energy of Helium-4.

I naturally borrow this classical notation for the surfaces  $E \times E$ ,  $M(E)$  and  $M^*(E)$ . I can now explicitly describe these surfaces in terms of 4-dimensional lattices derived from the 2-dimensional lattice  $L$ .

The square lattice  $L$  that defines  $E$  is just given by the integer lattice in the  $x, y$  variables. The dual lattice  $L'$  of face centres is  $L$  shifted by  $(1/2, 1/2)$ .

When we replace  $E$  by  $E \times E$  we must replace  $L$  by  $L \times L$ . All these have a distinguished origin or zero, making them abelian groups. But if we replace  $L$  by  $L'$  we get not a group but a coset of  $L$ , and the same applies when  $L \times L$  is replaced by  $L' \times L'$ . That is why, on symmetrizing, we get the fibration  $\text{Sym}^2(E) \rightarrow E_2$ , where  $E_2$  is not the Jacobian  $E_0$  but a coset.

Since  $L'$  is the dual of  $L$ ,  $L' \times L'$  is the dual of  $L \times L$ . There is a clear symmetrical relation between  $L \times L$  and  $L' \times L'$ , provided we ignore the origin and regard them

as affine lattices rather than vector lattices. This is necessary when we come to consider spin. Neither of the two symmetrized manifolds  $\text{Sym}^2(E) = M(E)$  or its dual  $\text{Sym}^2(E') = M(E')$  has a spin structure but the mutual double cover  $M^*(E) = M^*(E')$  does have one. This double cover treats the two on an equal footing and the switch between one and the other is induced by an affine motion and gives  $M(E)$  its flat affine connection. This can all be seen in terms of the classical coordinates  $q$  and  $z$ . As pointed out earlier,  $q^{1/2}$  is a spinor variable and is not invariant under the translation  $z \rightarrow z + 1$ ; it changes sign.

## 7 Conclusion

The main purpose of this paper was to provide a geometric model for Helium, explaining its main features, including the stability of Helium-4 and Helium-3. This we have now done using well-known abelian ideas.

The manifold  $M$  modelling static Helium-4, in its ground state, has a flat affine connection with non-zero torsion. Its fundamental group is the non-abelian quaternion group  $\Gamma$  of order 8.

The manifold  $M$  modelling static Helium-3 has a similar structure with fundamental group the non-abelian group  $\Sigma_3$  of order 6.

The model  $M = M(E)$  for Helium-4 is a complex surface constructed from the elliptic curve  $E$  with the symmetry of a square,  $y^2 + x^4 - 1 = 0$ .  $M(E)$  has, as universal covering space,  $M^*(E)$ , and this is essentially the surface given by the same equation over the quaternions  $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$  and the conjugates  $L'$  of  $L$  and  $E'$  of  $E$  arise from conjugation by  $j$ . Similar remarks apply to Helium-3.

The manifold  $M(E)$  modelling Helium-4 has cubical symmetry and other features of the physical atom. This should make it a useful model to study this remarkable and important element. At the theoretical level the model fits into the framework initiated with Greg Moore in [23].

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